Effective two-body approach to the hierarchical three-body problem

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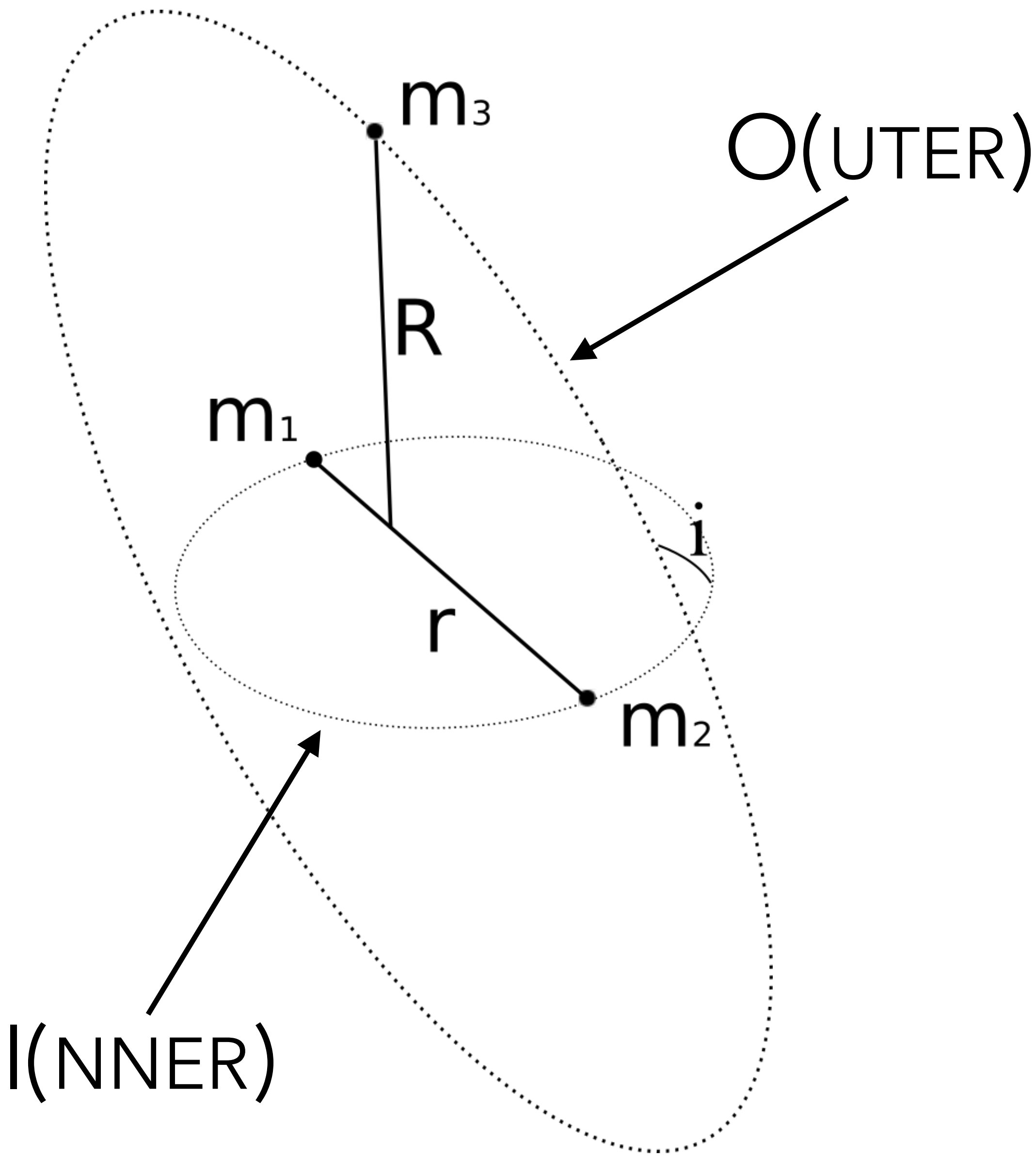
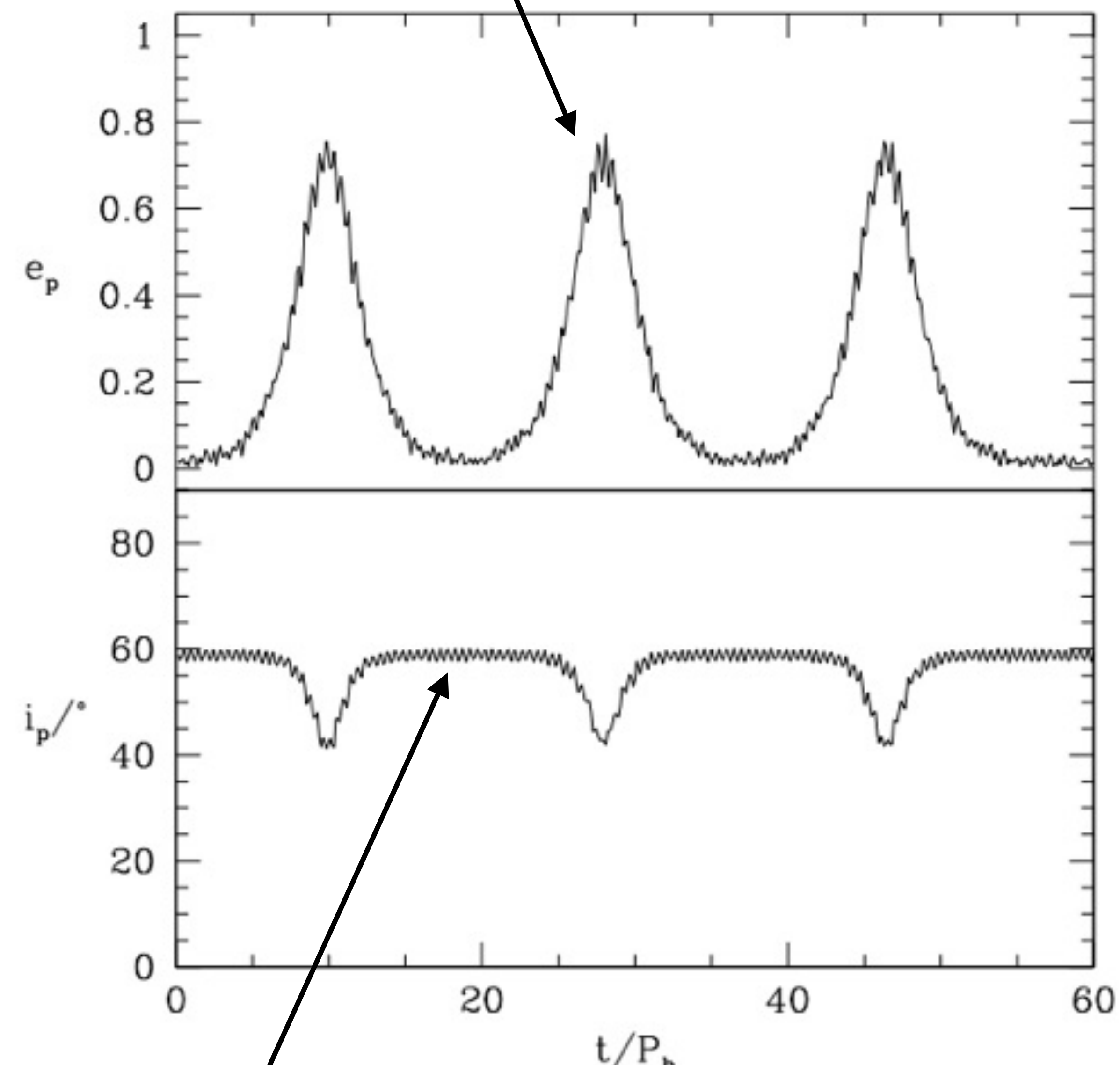
The Kozai-Lidov mechanism



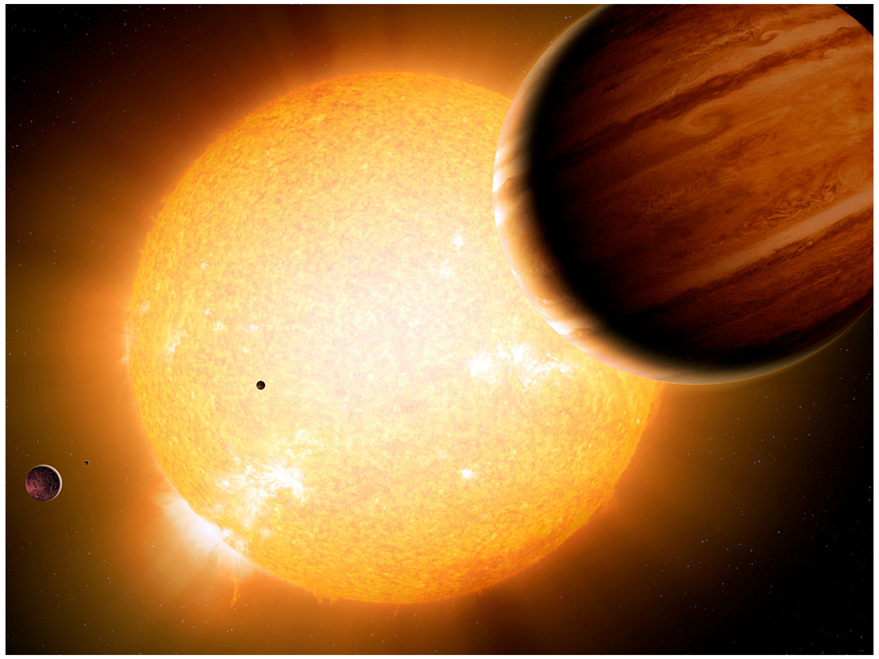
Long timescale $\frac{P_O}{P_I} P_O$



Fast oscillations $P_{I/O}$

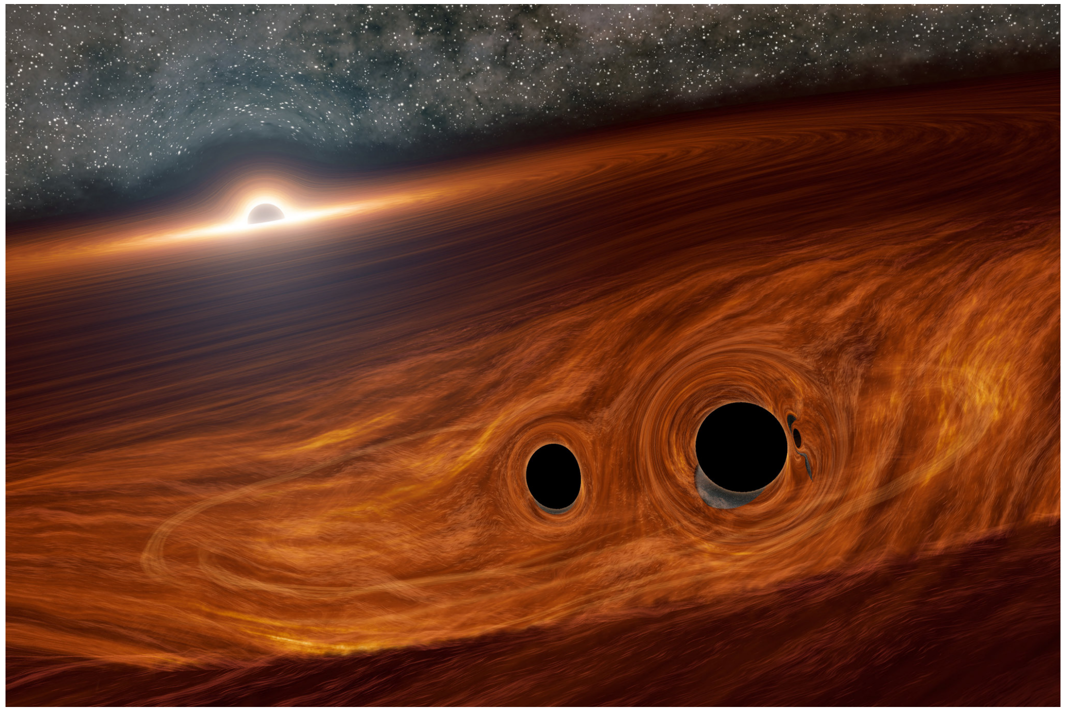
An incredible amount of applications !

- Earth's satellites (original Lidov-Kozai)

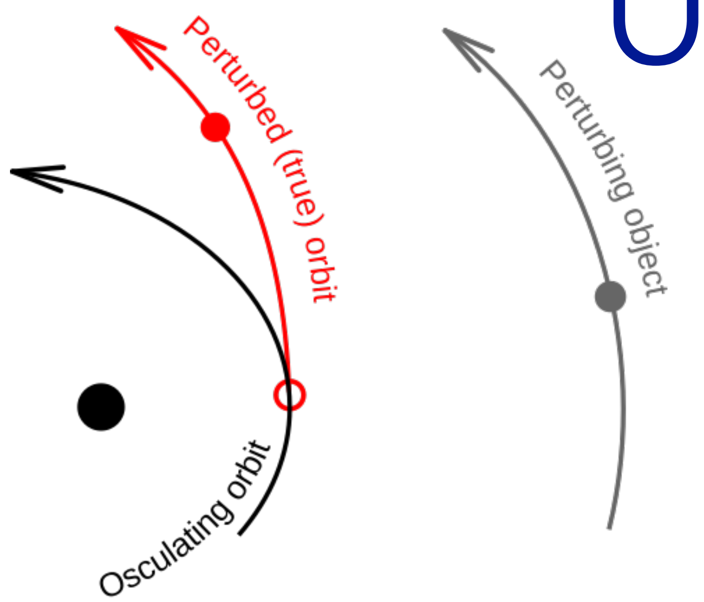


- Hot jupiter binaries

- Quick and eccentric mergers



USUAL TREATMENT



- Define the osculating orbit

- Write the 3-body Hamiltonian as an 'effective two-body' one:

$$H = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} + \frac{P_3^2}{2m_3} - \frac{G_N m_1 m_2}{|\mathbf{x}_1 - \mathbf{x}_2|} - \frac{G_N m_1 m_3}{|\mathbf{x}_1 - \mathbf{x}_3|} - \frac{G_N m_2 m_3}{|\mathbf{x}_2 - \mathbf{x}_3|}$$

$$= \left(\frac{p^2}{2\mu} - \frac{G_N m\mu}{r}\right) + \left(\frac{P^2}{2\mu'} - \frac{G_N M\mu'}{R}\right) - H'$$

$$m = m_1 + m_2$$
$$M = m_1 + m_2 + m_3$$
$$\mu = \frac{m_1 m_2}{m}$$
$$\mu' = \frac{m m_3}{M}$$

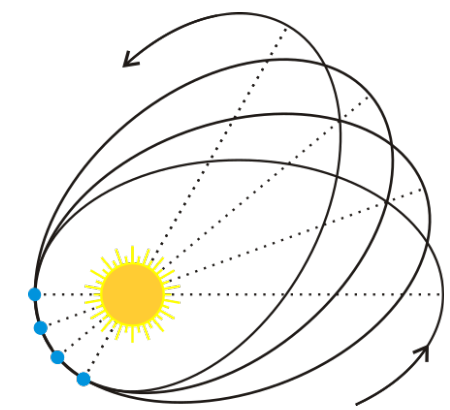
To quadrupole order :

$$H' = -\frac{G_N m_1 m_2 m_3}{2m}\frac{r^2}{R^3}\left(3\frac{(\mathbf{r}\cdot\mathbf{R})^2}{r^2 R^2} - 1\right)$$

USUAL TREATMENT

- Average the Hamiltonian over the short timescales of the two orbits

$$H' \;\rightarrow\; \langle\langle H' \rangle\rangle$$



- Finally, write the 'Lagrange planetary equations' (LPE):

$$n = \sqrt{\frac{G_N m}{a^3}} \qquad \tilde{H}' = \frac{H'}{\mu}$$

$$\dot{a} = \frac{2}{na}\frac{\partial \tilde{H}'}{\partial u}\,,$$

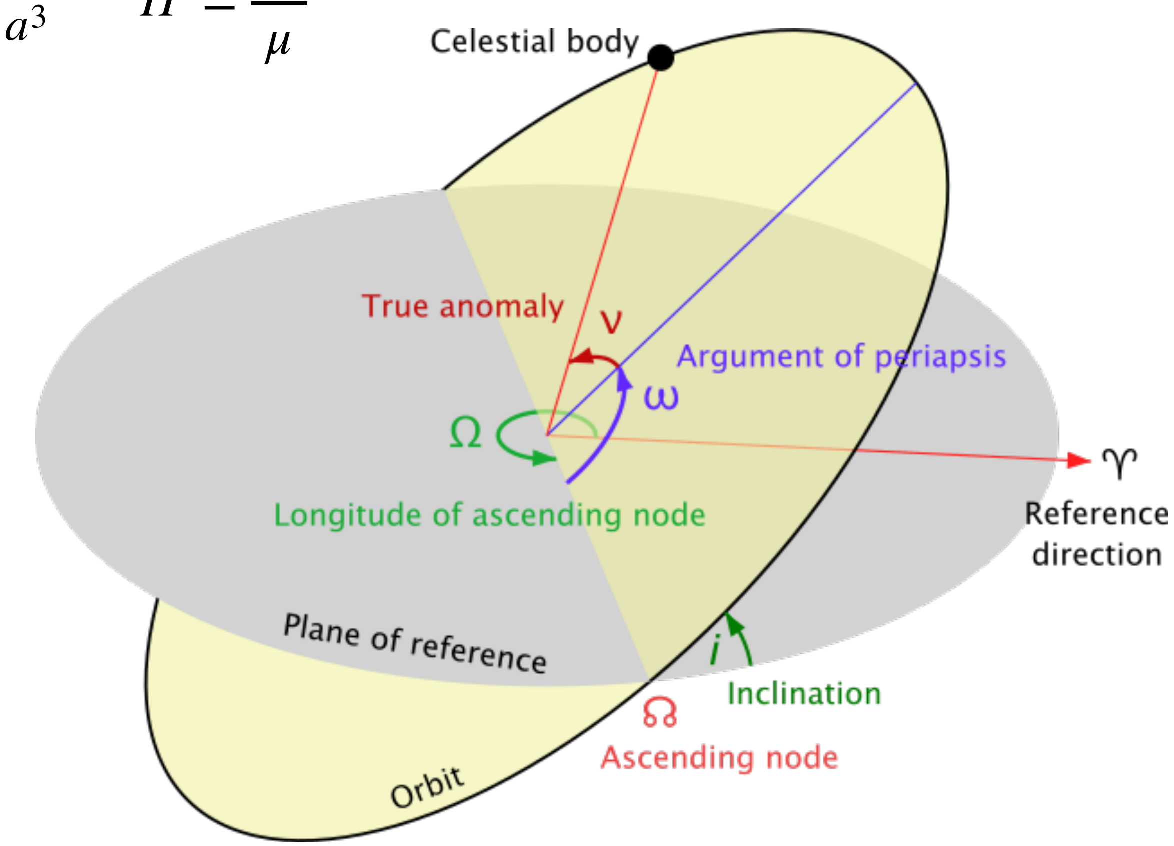
$$\dot{e} = -\frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial \tilde{H}'}{\partial \omega} + \frac{1-e^2}{na^2 e}\frac{\partial \tilde{H}'}{\partial u}\,,$$

$$\dot{\imath} = -\frac{1}{na^2\sqrt{1-e^2}\sin\imath}\frac{\partial \tilde{H}'}{\partial \Omega} + \frac{\cos\imath}{na^2\sqrt{1-e^2}\sin\imath}\frac{\partial \tilde{H}'}{\partial \omega}\,,$$

$$\dot{u} = n - \frac{2}{na}\frac{\partial \tilde{H}'}{\partial a} - \frac{1-e^2}{na^2 e}\frac{\partial \tilde{H}'}{\partial e}\,,$$

$$\dot{\omega} = \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial \tilde{H}'}{\partial e} - \frac{\cos\imath}{na^2\sqrt{1-e^2}\sin\imath}\frac{\partial \tilde{H}'}{\partial \imath}\,,$$

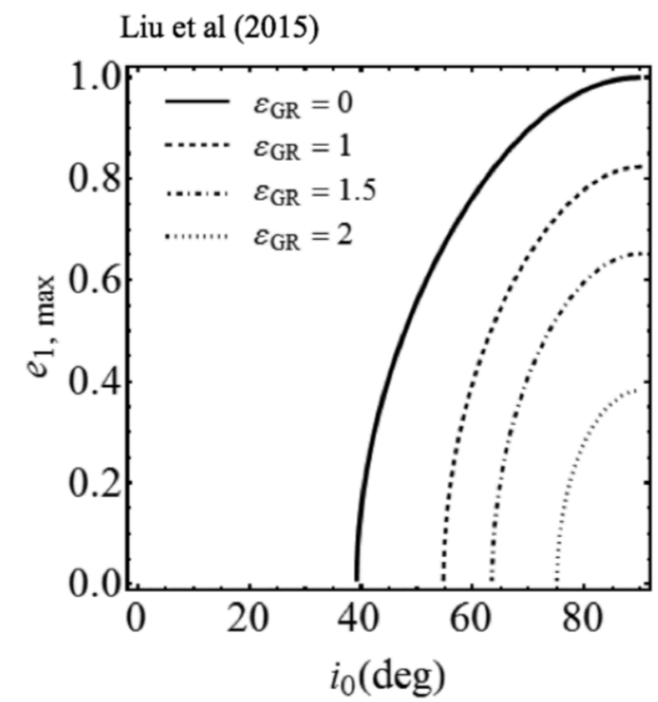
$$\dot{\Omega} = \frac{1}{na^2\sqrt{1-e^2}\sin\imath}\frac{\partial \tilde{H}'}{\partial \imath}\,,$$



RELATIVISTIC CORRECTIONS

- GR precession tends to suppress the eccentricity oscillations

$$\left.\frac{d\omega}{dt}\right|_{\rm PN} = \frac{3}{a(1-e^2)}\left(\frac{G_N m}{a}\right)^{3/2}$$



- What if the system as a whole is very relativistic ? Waveform ?

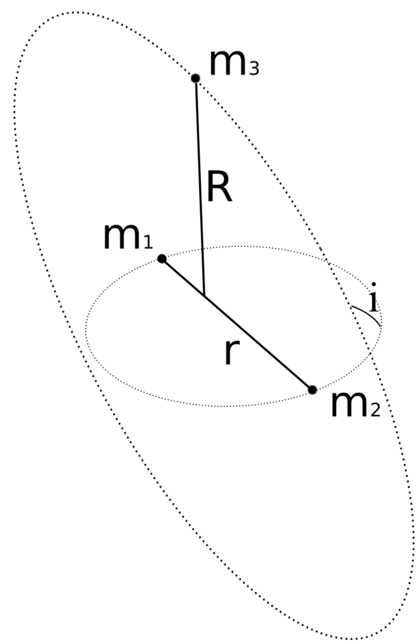
Very few studies ! Will (2014) Naoz (2013) Lim and Rodriguez (2020)

$$\boldsymbol{a}_a = -\sum_{b\neq a}\frac{Gm_b\boldsymbol{x}_{ab}}{r_{ab}^3} + \frac{1}{c^2}\sum_{b\neq a}\frac{Gm_b\boldsymbol{x}_{ab}}{r_{ab}^3}\left[4\frac{Gm_b}{r_{ab}} + 5\frac{Gm_a}{r_{ab}} + \sum_{c\neq a,b}\frac{Gm_c}{r_{bc}} + 4\sum_{c\neq a,b}\frac{Gm_c}{r_{ac}} - \frac{1}{2}\sum_{c\neq a,b}\frac{Gm_c}{r_{bc}^3}(\boldsymbol{x}_{ab}\cdot\boldsymbol{x}_{bc}) - v_a^2 + 4\boldsymbol{v}_a\cdot\boldsymbol{v}_b \right.$$
$$\left. - 2v_b^2 + \frac{3}{2}(\boldsymbol{v}_b\cdot\boldsymbol{n}_{ab})^2\right] - \frac{7}{2c^2}\sum_{b\neq a}\frac{Gm_b}{r_{ab}}\sum_{c\neq a,b}\frac{Gm_c\boldsymbol{x}_{bc}}{r_{bc}^3} + \frac{1}{c^2}\sum_{b\neq a}\frac{Gm_b}{r_{ab}^3}\boldsymbol{x}_{ab}\cdot(4\boldsymbol{v}_a - 3\boldsymbol{v}_b)(\boldsymbol{v}_a - \boldsymbol{v}_b),$$

The ‘cross terms’ can be very hard to obtain !

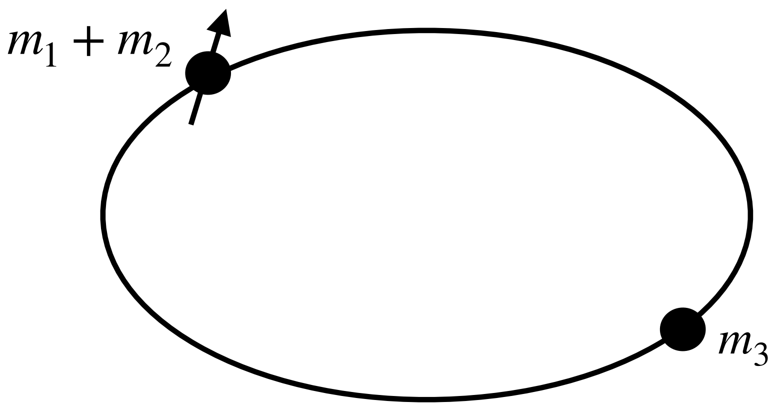
TAKE A STEP BACK

Couldn't we replace this system with a much simpler one ?



3-BODY WAVEFORM

$\Leftrightarrow$



2-BODY SPINNING TEMPLATE

Write down an EFT, to dipolar order:

$$\mathscr{L}_{\text{full}} = \sum_{A=1}^{3} - m_A \sqrt{-g_{\mu\nu} v_A^\mu v_A^\nu} \Rightarrow \mathscr{L}_{\text{EFT}} = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}u\, \mathscr{L}_{\text{full}} = -\mathscr{E}\sqrt{-g_{\mu\nu} V_{\text{CM}}^\mu V_{\text{CM}}^\nu} + \frac{1}{2} J_{\mu\nu} \Omega^{\mu\nu} - m_3 \sqrt{-g_{\mu\nu} v_3^\mu v_3^\nu}$$

Set up new power-counting rules, e.g:

$$V_{\text{CM}}^2 \sim \frac{G_N M}{A} = \frac{a}{A} \frac{G_N M}{a} \sim \frac{a}{A} v^2$$

The binary EFT

$$\mathcal{L}_{\rm EFT} = -\,\mathcal{E}\sqrt{-g_{\mu\nu}V^{\mu}_{\rm CM}V^{\nu}_{\rm CM}} + \frac{1}{2}J_{\mu\nu}\Omega^{\mu\nu}$$

To dipole order, the coupling of the inner binary to gravity is completely fixed by the equivalence principle

$$\mathcal{E} = m - \frac{G_N m}{2a}\,, \qquad J_{ij} = \epsilon_{ijk}J^k\,, \qquad \mathbf{J} = \sqrt{G_N m a(1-e^2)}\;\hat{\mathbf{j}}$$

$$\mathbf{X}_{\rm CM} = \frac{E_1}{E}\mathbf{x}_1 + \frac{E_2}{E}\mathbf{x}_2 \qquad \Leftrightarrow \qquad J_{0i} = 0$$

$$E_A = m_A + \frac{1}{2}m_A v_A^2 - \frac{G_N m_1 m_2}{2r}\,, \quad E = E_1 + E_2$$

The binary EFT

$$\mathscr{L}_{\text{EFT}} = -\mathscr{E}\sqrt{-g_{\mu\nu}V^{\mu}_{\text{CM}}V^{\nu}_{\text{CM}}} + \frac{1}{2}J_{\mu\nu}\Omega^{\mu\nu}$$

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But what is $\Omega^{\mu\nu}$?

$$\begin{aligned}
\dot a &= \frac{2}{n\,a}\frac{\partial \tilde H'}{\partial u}\,,\\
\dot e &= -\frac{\sqrt{1-e^2}}{n\,a^2 e}\frac{\partial \tilde H'}{\partial \omega} + \frac{1-e^2}{n\,a^2 e}\frac{\partial \tilde H'}{\partial u}\,,\\
\dot\iota &= -\frac{1}{n\,a^2\sqrt{1-e^2}\sin\iota}\frac{\partial \tilde H'}{\partial \Omega} + \frac{\cos\iota}{n\,a^2\sqrt{1-e^2}\sin\iota}\frac{\partial \tilde H'}{\partial \omega}\,,\\
\dot u &= n - \frac{2}{n\,a}\frac{\partial \tilde H'}{\partial a} - \frac{1-e^2}{n\,a^2 e}\frac{\partial \tilde H'}{\partial e}\,,\\
\dot\omega &= \frac{\sqrt{1-e^2}}{n\,a^2 e}\frac{\partial \tilde H'}{\partial e} - \frac{\cos\iota}{n\,a^2\sqrt{1-e^2}\sin\iota}\frac{\partial \tilde H'}{\partial \iota}\,,\\
\dot\Omega &= \frac{1}{n\,a^2\sqrt{1-e^2}\sin\iota}\frac{\partial \tilde H'}{\partial \iota}\,,
\end{aligned}
\qquad \Leftrightarrow \qquad \mathscr{L}_{\text{EFT}} = \mathbf{J}\cdot\boldsymbol{\Omega}$$

The LPE are just a spin kinetic term! The angular velocity is:

$$\boldsymbol{\Omega} = \hat{\mathbf{e}} \times \dot{\hat{\mathbf{e}}}$$

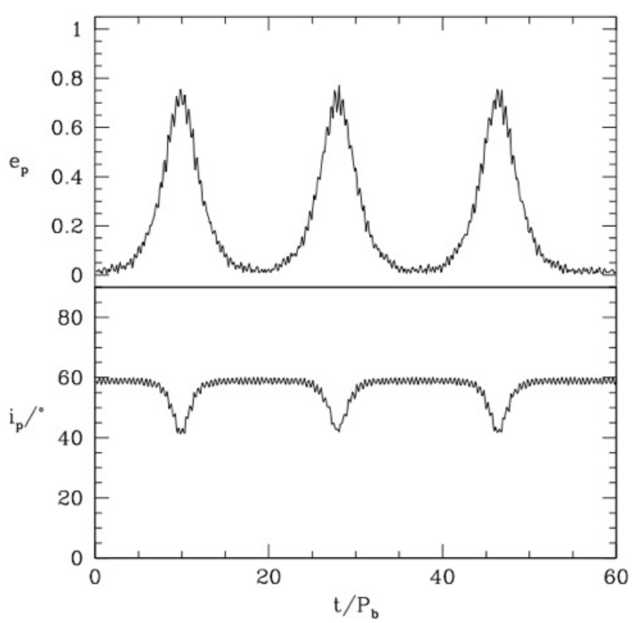
$\hat{\mathbf{e}} \equiv$ Unit Runge-Lenz vector

QUADRUPOLAR ORDER

At 1PN quadrupole order there are three new operators:

$$\mathcal{O}_1 = \frac{1}{2}\int \mathrm{d}\tau\, Q^{ij}E_{ij} \qquad \mathcal{O}_2 = -\frac{1}{3}\int \mathrm{d}\tau\, I^{ij}B_{ij} \qquad \mathcal{O}_3 = -C_{S^2}\int \mathrm{d}\tau\, S^iS^jE_{ij}$$

C_{S^2} should be computed by matching. Physical interpretation :



$$e = \langle e \rangle + \delta e$$

Low-energy DOF, assumed constant in the averaging procedure

High-energy fluctuations, to be integrated out

Conclusions

- The EFT treatment is ideally suited for the hierarchical 3-body pb
- Symmetries allow to guess the form of the 'cross terms' without a single computation; this was obscure in conventional treatment
- A nice explicit example of a composite point-particle in NRGR
- Higher-order matching computations and waveform modelling now more tractable