Unruh effect

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Abstract

The Unruh effect is the temperature feeled by an accelerating observer in Minkowski space. We will derive as simply as possible how this temperature emerges.

An accelerating observer in 2D Minkowski space moves along an hyperbola represented in Figure 1 (for different accelerations).

This hyperbola is parametrized by :

$$\begin{cases} t(s) &= \frac{1}{a}\sinh(sa) \\ x(s) &= \frac{1}{a}\cosh(sa) \end{cases} \tag{1}$$

The acceleration of this body is $a^\alpha a_\alpha = a^2$. We can switch to coordinates adapted to this observer, the Rindler coordinates :

$$\begin{cases} t &= e^z \sinh(\tau) \\ x &= e^z \cosh(\tau) \end{cases} \tag{2}$$

e^z parametrizes which hyperbola the observer sit on (we have now normalized the acceleration such that $a = 1$). If we denote $x^\pm = x \pm t$, we have $x^+ = e^{z+\tau}$ and $x^- = e^{z-\tau}$. The metric in Rindler coordinates is conformally equivalent to Minkowski metric :

$$ds^2 = -dt^2 + dx^2 = e^{2z}(-d\tau^2 + dz^2) \tag{3}$$

Now consider a quantum massless scalar field in both Minkowski and Rindler coordinates. As usual for quantum fields in curved space, there is an ambiguity in determining the vacuum of this field. The action for this field writes :

$$S = \frac{1}{2}\int d^2x \sqrt{-g} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{4}$$

which takes the same form in both coordinates :

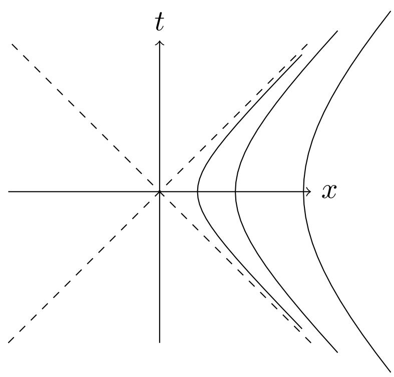

Figure 1: A family of Rindler observers

$$S = \frac{1}{2}\int dxdt(-(\partial_t\phi)^2 + (\partial_x\phi)^2) = \frac{1}{2}\int dzd\tau(-(\partial_\tau\phi)^2 + (\partial_z\phi)^2) \tag{5}$$

Let's now go in Fourier space :

$$\phi^M(t,k) = \int \mathrm{d}x e^{ikx}\phi^M(t,x) \tag{6}$$

$$\phi^R(\tau,K) = \int \mathrm{d}z e^{iKz}\phi^R(\tau,z) \tag{7}$$

I have introduced two different notations to make explicit the distinction between the two different coordinates systems. These two functions obey the differential equations (obtained from minimizing the action) :

$$\partial_t^2\phi^M + k^2\phi^M = 0 \tag{8}$$

$$\partial_\tau^2\phi^R + K^2\phi^R = 0 \tag{9}$$

The solutions of these equations are waves $\phi^M_{cl} = \frac{\alpha_k}{\sqrt{2k}}e^{-ikt} + \frac{\beta_k}{\sqrt{2k}}e^{ikt}$ (resp. $\phi^R_{cl} = \frac{\alpha_K}{\sqrt{2K}}e^{-iK\tau} + \frac{\beta_K}{\sqrt{2K}}e^{iK\tau}$). Now, a simple computation shows that to minimize the classical value of the Hamiltonian $H = \frac{1}{2}(\dot{\tilde{\phi}}^2_{cl} + k^2\phi^2_{cl})$, one should take $\alpha_k = 1$ and $\beta_k = 0$. Let's show this.

The temporal mean value of the Hamiltonian is $\frac{1}{2}\alpha\beta k$. In addition, we want to write $\tilde{\phi}$ as a quantum field, so it must obey the uncertainty principle. More precisely, if we write $\tilde{\phi} = \tilde{\phi}_{cl}a + \tilde{\phi}^*_{cl}a^\dagger$ and impose that $[\tilde{\phi}, \dot{\tilde{\phi}}] = i$ knowing $[a, a^\dagger] = 1$, then the coefficients must obey $\alpha^2 - \beta^2 = 1$. The only solution minimizing the (positive) Hamiltonian is $\alpha = 1$, $\beta = 0$.

We now want to quantize this field. There is an ambuguity in the choice of vacuum because of the two different coordinate systems :

$$\phi^R(\tau,K) = \phi^R_{cl}(\tau,K)a_R + \phi^{R*}_{cl}(\tau,K)a^\dagger_R \quad , \quad a_R\left|0\right\rangle_R = 0 \tag{10}$$

$$= \phi^M_{cl}(\tau,K)a_M + \phi^{M*}_{cl}(\tau,K)a^\dagger_M \quad , \quad a_M\left|0\right\rangle_M = 0 \tag{11}$$

where we have taken the classical value corresponding to the minimum energy $\alpha = 1$, $\beta = 0$.

The question is now : is the Minkowski vacuum in Rindler coordinates really a vacuum ? More precisely, how many Rindler particles are there in the Minkowski vacuum ? To know this we should calculate ${}_M\langle 0|\, a^\dagger_R a_R \,|0\rangle_M$. We will show that this quantity is proportional to $\frac{1}{e^{\beta K}-1}$ which is a bose distribution of inverse temperature β, showing that this accelerated quantum field feels this Unruh temperature (one can also do a similar treatment for fermions). In natural units, we will show that $\beta = 2\pi$, corresponding to a temperature when reintroducing physical constants (and an acceleration a) :

$$T_{Unruh} = \frac{a\hbar}{2\pi k_B c} \tag{12}$$

Let's do this job. The first step is to express $\phi^M_{cl}(\tau,K)$ in terms of $\phi^R_{cl}(\tau,K) = \frac{e^{-iK\tau}}{\sqrt{2K}}$. Since $\phi^M_{cl}(t,x) = \int \frac{dk}{2\pi}e^{-ikx}\phi^M_{cl}(t,k) = \int \frac{dk}{2\pi\sqrt{2k}}e^{-ik(x-t)}$, one has $\phi^M_{cl}(\tau,z) = \int \frac{dk}{2\pi\sqrt{2k}}e^{-ike^{z-\tau}}$, so that, for a single frequency k :

$$\begin{aligned}
\phi^M_{cl}(\tau,K) &\propto \int \mathrm{d}z e^{iKz}e^{-ike^{z-\tau}} \\
(u = ike^{-\tau}e^z) \quad &= \int_0^\infty \frac{\mathrm{d}u}{u}\left(\frac{-iu}{ke^{-\tau}}\right)^{iK} e^{-u} \\
&= e^{K\frac{\pi}{2}}e^{iK\tau}e^{-ik\log k}\Gamma\left(-iK\right) \\
&\equiv \gamma\phi^{R*}_{cl}(\tau,K)
\end{aligned} \tag{13}$$

with $\gamma \propto e^{K\frac{\pi}{2}} e^{-ik\log k} \Gamma\left(-iK\right)$.

Now we can relate a_M and a_R using equation 11 :

$$\phi^R(\tau, K) = \gamma \phi_{cl}^{R*}(\tau, K) a_M + \gamma^* \phi_{cl}^R(\tau, K) a_M^\dagger \tag{14}$$

from which it follows that $a_R = \gamma^* a_M^\dagger$ (γ is called a bogoliubov coefficient). So ${}_M\langle 0|\, a_R^\dagger a_R \,|0\rangle_M = |\gamma|^2 \; {}_M\langle 0|\, a_M a_M^\dagger \,|0\rangle_M = |\gamma|^2$. Using the identity $|\Gamma\left(-iK\right)|^2 = \frac{\pi}{K \sinh \pi K}$, one can easily derive that :

$$|\gamma|^2 \propto \frac{1}{e^{2\pi K} - 1} \tag{15}$$

which shows the previous result.