An EFT for the two-body problem

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Introduction: • We consider a bound system of two black holes (BH) or neutron stars (NS) ($m \sim M_\odot \sim 10^{30}$ kg) emitting gravitational waves (GW)

Some crude order-of-magnitude estimates:

At large distance, gravitational energy $E_{\text{Newton}} \sim \frac{Gm^2}{r} \sim m v^2$

Virial theorem : $\boxed{v^2 \sim \frac{Gm}{r} \ll 1}$

kinetic energy ← gravitational strength

v^2 is a small parameter and this is a good starting point to build an EFT! We will carry out an expansion in v^2.

This corresponds to the inspiral phase of the dynamics. Our calculations will lose accuracy when the BH/NS are too close. Typically the computations that I will present will work well up to $v^2 \sim 0.3$.

Why is this phase called inspiral? Because the BH/NS lose energy in GW and get closer!

Quadrupole formula : $\boxed{P = \frac{32}{5G} v^{10}}$ dissipated power in GW to lowest order in v

Let's evaluate the duration of such an event:

$E \sim m v^2$ $P \sim \frac{v^{10}}{G}$ & balance equation : $\frac{dE}{dt} = P \Rightarrow \frac{dv}{dt} \sim \frac{v^9}{Gm}$

Number of cycles/orbits : $\phi = 2 \int \omega \, dt \sim$

For a Keplerian orbit $\omega \sim \frac{v}{r} \sim \frac{v^3}{Gm}$

$$\Rightarrow \phi \sim \int \frac{v^3}{Gm} \frac{dv}{v^9} Gm \sim \frac{1}{v_{in}^5} - \frac{1}{v_f^5}$$

In observatories we can observe up to $v_{in} \sim 0.1$

This means that we can observe up to 10^5 orbits!

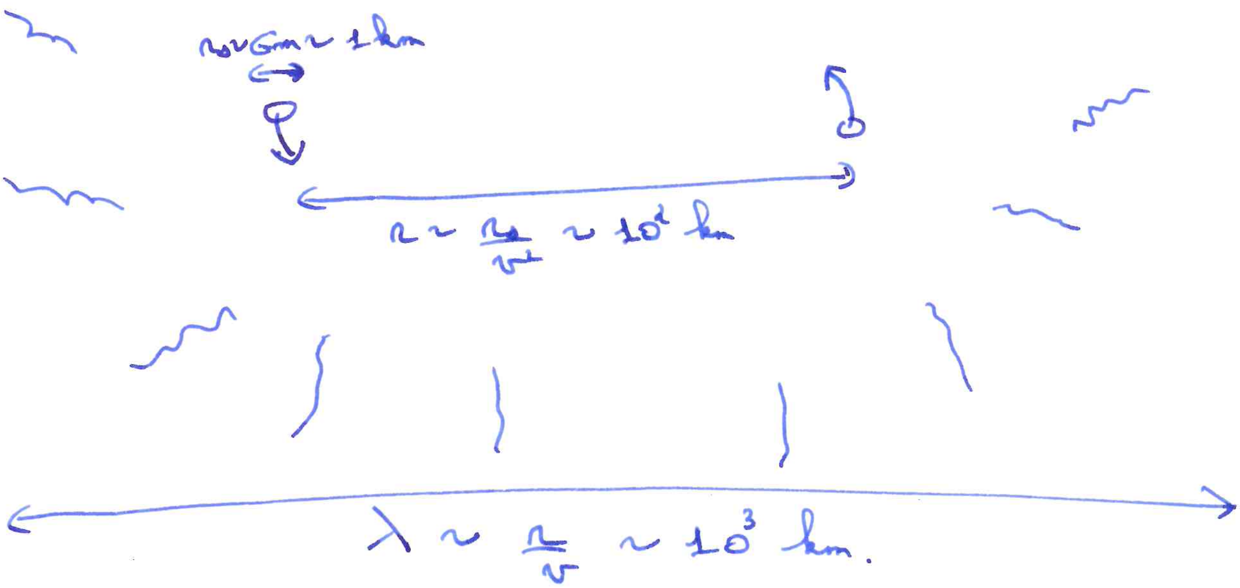
Since we are sensitive to ± 1 orbit, we need to compute ϕ with great accuracy. This means taking into account all the relativistic corrections up to a high order in perturbation theory.

On the other hand you know a lot of particle physics methods adapted to precision computation. Is it possible to reuse them here?

$\Rightarrow$ YES: Non-Relativistic General Relativity (NRGR).

<u>Schematic description of the system and the different lengthscales:</u>

<u>Reference</u>: Goldberger's lectures notes hep-ph/0701129



I Building the EFT action for a 2-body system

Degrees of freedom: • Metric $g_{\mu\nu}(x^\mu) = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{m_P}$

• Worldlines of the 2 BH/NS: $x_A^\mu(\lambda_A)$ $\lambda_A \equiv$ affine parameter

Symmetries: • ~~Reparam~~ Diffeomorphism invariance $x^\mu \to \tilde{x}^\mu(x^\alpha)$

• Reparametrization of affine parameters $\lambda_A \to \tilde{\lambda}_A(\lambda_A)$ $A = 1, 2$

We will also add:

• SO(3) symmetry: objects do not have (traceless) multipole moments.

Gravitational action: $\boxed{S = S_{EH} = \frac{m_P^2}{2} \int d^4x \sqrt{-g}\, R}$

We could also imagine to have terms like $m_P^2 \int d^4x \sqrt{-g}\, \underbrace{\frac{R^2}{m_P^2}}_{\text{only mass scale}}$

However such operators are Planck-suppressed:

for GWs $E \sim (10^3\,\text{km})^{-1} \sim 1\,\text{eV} \Rightarrow \frac{E^2}{m_P^2} \sim 10^{-56}$!

BH/NS action: We model them as point-particles

The proper time is gauge-invariant: $\boxed{d\tau_A^2 = -g_{\mu\nu}\, dx_A^\mu dx_A^\nu}$

$\Rightarrow \boxed{S_{pp,A} = -m_A \int d\tau}$

Exercise: Vary the point-particle action wrt x_A^μ and find the geodesic equation: $\frac{d^2 x_A^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{dx_A^\nu}{d\tau} \frac{dx_A^\rho}{d\tau} = 0$.

Nonminimal operators: We could also add other worldline operators: $\mathcal{O}_1 = c_1 \int d\tau\, R$ $\mathcal{O}_2 = c_2 \int d\tau\, R_{\mu\nu} v^\mu v^\nu$

Q: What ~~do~~ these operator correspond to physically? $\left(v^\mu \equiv \frac{dx^\mu}{d\tau}\right)$

A: Deviation from geodesic equation $\Rightarrow$ finite-size effects

In fact, these operators do not contribute at lowest order.

This is because in an EFT, you are allowed to use the equations of motion perturbatively in the action.

Here: $R_{\mu\nu} = 0 \Rightarrow$ you can remove any quantity depending on $R_{\mu\nu}$ in the action.

There remains some ~~op~~ nontrivial operators involving the "electric" and "magnetic" parts of the Riemann tensor:

$E_{\mu\nu} = R_{\mu\alpha\nu\beta} v^\alpha v^\beta \qquad B_{\mu\nu} = \varepsilon_{\mu\alpha\beta\rho} v^\rho R^{\alpha\beta}{}_{\rho\nu} v^\rho$

$\Rightarrow \boxed{O_3 = c_E \int d\tau\, E_{\mu\nu} E^{\mu\nu}} \qquad \boxed{O_4 = c_B \int d\tau\, B_{\mu\nu} B^{\mu\nu}}$

These are the lowest-order finite-size effects. I will discuss them again at the end of the course.

II Integrating out $\gamma_{\mu\nu}$

What we are ultimately interested in is the trajectory of the 2 BH/NS $x_A^\mu(\lambda)$.

They can be obtained by computing the effective action

Formally: $\boxed{e^{iS_{eff}[x_A^\mu]} = \int \mathcal{D}h_{\mu\nu}\, e^{iS[x_A^\mu;\, \gamma_{\mu\nu}]}}$ (also denoted as Z)

where I recall that the action is: $S = \frac{m_P^2}{2}\int d^4x \sqrt{g} R - \sum_{A=1,2} m_A \int d\tau_A$

S_{eff} contains all the information we need about the dynamics of BH/NS

If no energy leaves the system:

$$S_{eff}[x_A^\mu] = \int dt\, L[\vec{x}_A, \vec{v}_A].$$

However there will be definitely some energy leaving in GWs!

This can be related to $\mathrm{Im}(S_{eff})$ in the following way.

In the "in-out" formalism, one has: $Z = e^{iS_{eff}} = \langle 0_+ | 0_- \rangle$ overlap between initial & final states.

$\Rightarrow |\langle 0_+|0_-\rangle|^2 = e^{-2\,\mathrm{Im}(S_{eff})} \simeq 1 - 2\,\mathrm{Im}\, S_{eff}$

$\uparrow$ if small interaction time

On the other hand:

$|\langle 0_+ | 0_- \rangle|^2 = 1 - p\,T$ where $p = \int dE\, d\Omega \frac{d^2\Gamma}{dE\, d\Omega}$ particle production probability rate

T = time of interaction.

$$\Rightarrow \boxed{\operatorname{Im} S_{eff} = \frac{T}{2} \int dE\, d\Omega \frac{d^2\Gamma}{dE\, d\Omega}}$$

Once we know $\operatorname{Im} S_{eff}$, this can be used to compute the dissipated power $P = \int E \frac{d^2\Gamma}{dE\, d\Omega} dE\, d\Omega$

<u>Perturbative expansion</u>: We split $g_{\mu\nu} = \eta_{\mu\nu} + \underbrace{\frac{h_{\mu\nu}}{m_p}}_{\ll 1}$

A complicated computation gives the action expanded to quadratic order in $h_{\mu\nu}$:

$$\boxed{S_{EH}^{(2)} = -\frac{1}{8} \int d^4x \left[(\partial_\mu h_{\nu\rho})^2 - \frac{1}{2} (\partial_\mu h^\alpha{}_\alpha)^2 \right].}$$

<u>WARNING</u>: this computation has been done using a specific gauge-fixing term - see the Reference for more details. This corresponds to choose the so-called "harmonic coordinates", satisfying $g^{\mu\nu} \nabla_\mu \nabla_\nu x^\alpha = 0$

In Fourier-space:

$$S_{EH}^{(2)} = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} k^2 h_{\mu\nu}(k) T^{\mu\nu;\alpha\beta} h_{\alpha\beta}(-k).$$

with $T^{\mu\nu;\alpha\beta} = \eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta}$

Define the inverse of $T^{\mu\nu;\alpha\beta}$ by:

$$P_{\mu\nu;\rho\sigma} T^{\rho\sigma;\alpha\beta} = \frac{1}{2}\left(\delta_\mu^\alpha \delta_\nu^\beta + \delta_\mu^\beta \delta_\nu^\alpha\right)$$ identity on symmetric rank-2 tensors.

<u>Exercise</u>: Show that $\boxed{P_{\mu\nu;\alpha\beta} = 2\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right)}$ does the job.

- The <u>propagator</u> of h is given by:

$$\boxed{\begin{aligned} \langle h_{\mu\nu}(x)\, h_{\alpha\beta}(x') \rangle &= D_F(x-x')\, P_{\mu\nu\alpha\beta} \\ D_F(x-x') &= \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 - i\varepsilon} e^{-ik(x-x')} \end{aligned}}$$ .....

At next order in perturbation theory there is also a cubic vertex in the EH action, schematically:

$$S_{EH} \sim \int d^4x \, \partial^2 h^3 .$$

The full expression is very long and complicated, and will not be used in these notes.

Finally there remains to expand the point-particle action:

$$S_{pp,A} = -m_A \int d\tau_A = -m_A \int dt \sqrt{-g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}} \qquad \frac{dx^\mu}{dt} = (1, \vec{v})$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{m_p}$$

$$= -m_A \int dt \sqrt{1 - v^2 - h_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}}$$

$$\simeq -m_A \int dt \Big[1 - \frac{v^2}{2} - \frac{v^4}{8} + \mathcal{O}(v^6)$$

$$- \frac{h_{00}}{2m_p} - \frac{h_{0i}}{m_p} v^i - \frac{h_{ij} v^i v^j}{2 m_p} - h_{00} \frac{v^2}{4} + \mathcal{O}(h v^3)$$

$$- \frac{h_{00}^2}{8 m_p^2} + \mathcal{O}(h^2 v) \Big]$$

The reason why I kept only these vertex will become clearer later on.

From such a propagator and vertex one would like to obtain some definite power-counting rules in v which tell you how many vertex you have to include in the action. However this is not possible at the moment. This comes from the fact that the propagator does not scale homogeneously in v.

To see this, consider a graviton giving rise to the Newtonian potential. Necessarily, it has to vary on spatial lengthscales $|\vec{k}| \sim 1/r$. On the other hand the Newtonian frequency of the system is $k^0 = \omega \simeq \frac{v}{r} \ll |\vec{k}|$.

But this also mean that such a graviton can never be on-shell and propagate to give gravitational radiation!

To remedy to this, we split $h_{\mu\nu}$ into 2 modes:

$$h_{\mu\nu} = H_{\mu\nu} + \bar{h}_{\mu\nu}$$

potential mode $\Leftrightarrow$ conservative $k^\mu \sim \left(\frac{v}{r}, \frac{1}{r}\right)$

radiation mode $\Leftrightarrow$ dissipative $k^\mu \sim \left(\frac{v}{r}, \frac{v}{r}\right)$

This mean that, in our EFT approach, we compute the path-integral in 2 steps by first integrating out the highest energy modes:

$$e^{iS_{eff}} = \int \mathcal{D}\bar{h}_{\mu\nu} \int \mathcal{D}H_{\mu\nu}\, e^{iS[g_{\mu\nu}, x_i^\mu]}.$$

The propagator for the potential modes simplifies to: (at lowest order)

$$\boxed{\langle H_{\mu\nu}(x)\, H_{\alpha\beta}(x')\rangle = -i\,\delta(t-t')\int \frac{d^3k}{(2\pi)^3}\, \frac{e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}}{\vec{k}^2}\, P_{\mu\nu;\alpha\beta}}$$

There are also v^2 corrections to the propagator: by expanding to $O(k_0^2)$,

$$\langle H_{\mu\nu}(x)\, H_{\alpha\beta}(x')\rangle_{v^2} = -i\,\frac{d^2}{dt\,dt'}\,\delta(t-t')\int \frac{d^3k}{(2\pi)^3}\, \frac{e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}}{\vec{k}^4}\, P_{\mu\nu,\alpha\beta}$$

--X--

<u>Exercise</u>: Compute the integral defining the propagator and show:

$$\int \frac{d^3k}{(2\pi)^3}\,\frac{e^{-i\vec{k}\cdot\vec{r}}}{\vec{k}^2} = \frac{1}{4\pi r}$$

Power-counting and Feynman rules

We can now setup the power-counting rules (PCR) of the theory.

$$H \sim \sqrt{\text{propagator}} \sim \sqrt{\frac{v}{r}\cdot\frac{1}{r}} \sim \frac{\sqrt{v}}{r}$$

(I have used $t \sim \frac{1}{k^0} \sim \frac{r}{v}$ since $k^0 \sim \frac{1}{t} \sim \frac{v}{r}$ for both modes)

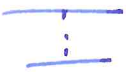
$$h \sim \sqrt{\left(\frac{v}{r}\right)^4\left(\frac{r}{v}\right)^2} \sim \frac{v}{r}$$

$$\frac{m}{m_p}\int dt\, H \sim \frac{m}{m_p}\cdot\frac{r}{v}\,\frac{\sqrt{v}}{r} \sim \frac{m}{m_p\sqrt{v}}$$

To find the scaling of $\frac{m}{m_p}$, remember the virial theorem:

$$v^2 \sim \frac{m}{m_p^2 r} \Rightarrow \frac{m}{m_p} \sim \sqrt{m v^2 r} \sim \sqrt{Lv} \quad \text{where } L = mvr$$
$= $ <u>angular momentum</u>

We obtain the scaling:

 $\sim \sqrt{L}$ so that  $\sim L$. We will compute this diagram in a moment.

<u>Exercise</u>: show that:

$$m\int dt\, v^2 \sim L \quad , \quad m\int dt\, v^4 \sim L v^2$$

$$\frac{m}{m_P}\int dt\, H_{0i} v^i \sim \sqrt{L}\, v \quad , \quad \frac{m}{m_P}\int dt\, H_{ij} v^i v^j \sim \sqrt{L}\, v^2$$

$$\frac{m}{m_P}\int dt\, H_{00}^2 \sim v^2$$

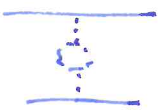
$$\int d^4x\, \frac{\partial^2 H^3}{m_P} \sim \frac{v^2}{\sqrt{L}}$$

The Feynman rules are as follows:

- At a given order in v, draw all the diagrams that remain connected once the worldlines of the point-particles are removed (this is because the point-particules are treated as external classical sources for $h_{\mu\nu}$, and the product of disconnected diagrams have been factorized by the exponential definition of the effective action $e^{iS_{eff}}$, called also the "connected diagrams generating function").
- For each vertex, multiply by $i \times$ the previous expressions obtained
- Contract all gravitons with their propagators
- Divide by the symmetry factor of the diagram

<u>Just a word about loops</u>:

One could be tempted to consider diagrams like

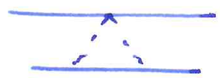


One can show (cf App. A of hep-th/0409156) that a diagram with ℓ gravitons loops scales as $L^{1-\ell}$ $\quad L = mrv$ angular momentum

Restoring the correct units: $L = \frac{m r v}{\hbar} \sim 10^{70}$!

So ~~loops~~ graviton loops are completely negligible.

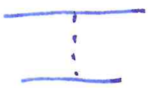
One could also think that this is a loop:



In fact this is not, since the point-particles act as a fixed background source and do not propagate.

$\boxed{\text{III The conservative dynamics up to 1 PN order.}}$

Let's now compute exactly the most ~~simple~~ 1-graviton exchange diagram.



$$= i\frac{m_1}{2m_P}\int dt_1 \, \frac{i m_2}{2m_P}\int dt_2 \, \underbrace{\langle H_{00}(t_1, x_1(t_1)) \, H_{00}(t_2, x_2(t_2))\rangle}_{= -i\delta(t_1-t_2)\frac{P_{00;00}}{4\pi|x_1-x_2|}}$$

$$= \frac{i m_1 m_2}{16\pi m_P^2}\int dt \, \frac{1}{r} \times P_{00;00}$$

$$= i\int dt \, \frac{G m_1 m_2}{r} \quad \text{Newton's energy.}$$

So at order $\sqrt{L}$ the effective action is: $S_{eff} = i\int dt \left[m_1\frac{v_1^2}{2} + m_2\frac{v_2^2}{2} + \frac{G m_1 m_2}{r} \right]$

One could also compute a diagram like this one:

It turns out that it is divergent and you have to use a regularisation scheme to compute a divergent integral: $\int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2}$

In dimensional regularisation, such power-law divergences are just set to zero.

You could (rightly) have the feeling that something is being swept under the rug. There can indeed be another nice interpretation of this diagram in terms of the equivalence principle.

To see this, let's adopt a cutoff regularization: $k < K \sim \frac{1}{R}$.
R has the physical interpretation of being the size of the point-particle (the point at which our EFT breaks down because new physics related to the size of the point-particles matter)
for $k \sim 1/R$

Then one can compute:

$$= \int dt \, i \frac{m_1^2}{4 m_p^2} P_{00;00} \int \frac{d^3k}{(2\pi)^3 k^2} = \int dt \frac{2iG m_1^2}{\pi R}$$

But this is similar to the gravitational self-energy of m_1:

$$U = -G \int \frac{d^3x \, d^3y}{|x-y|} \rho(x)\rho(y) = -3 \frac{G m_1^2}{5R}$$

In the effective action, such a term will just renormalize the bare mass: $-m_1 \int dt \longrightarrow -(m_1 + U)\int dt$.

So we get that the mass of a ~~he~~ heavy object should include its self-gravitational energy! This is called the strong equivalence principle: all objects fall in the same way in an external gravitational field, whatever their composition and gravitational energy.

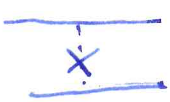
A 1PN diagram: Let's compute the following diagram:

$$= \frac{1}{2!} \frac{i m_1}{8 m_p^2} \int dt_1 \frac{i m_2}{2 m_p} \int dt_2 \frac{i m_2}{2 m_p} \int dt_2' \times \langle H_{00}(t_1, x_1(t_1))^2 H_{00}(t_2, x_2(t_2)) H_{00}(t_2', x_2(t_2')) \rangle$$

$$= \frac{i m_1 m_2^2 P_{00;00}^2}{2^5 m_p^4} \int dt \left(\frac{1}{4\pi r}\right)^2$$ (I have dropped the divergences)

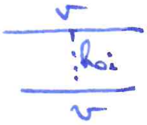
$$= i \int dt \frac{m_1 m_2^2 G^2}{2 r^2}$$

Exercise: Show that:

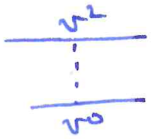


$$= i \int dt \frac{G m_1 m_2}{2r} \left(\vec{v_1} \cdot \vec{v_2} - \frac{(\vec{v_1}\cdot\vec{r})(\vec{v_2}\cdot\vec{r})}{r^2} \right) \qquad \vec{r} = \vec{x_1} - \vec{x_2}$$

Exercise (continued)

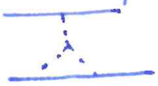


$$= -4i\int dt\, \frac{G m_1 m_2}{r}\, \vec{v}_1\cdot\vec{v}_2$$



$$= \frac{3}{2}\, i\int dt\, \frac{G m_1 m_2}{r}\left(v_1^2+v_2^2\right)$$

Finally I give you the last diagram, which is quite complicated to compute:



$$= -\, i\int dt\, \frac{G^2 m_1 m_2^2}{r^2}$$

Summing all the diagrams (also the symmetric $1\leftrightarrow 2$), we get to the following $\mathcal{L}v^2$ effective action:

$$S_{eff,\mathcal{L}v^2} = \int dt\Big[\frac{1}{8} m_1 v_1^4 + \frac{1}{8} m_2 v_2^4 + \frac{G m_1 m_2}{2r}\Big(3(v_1^2+v_2^2) - 7\,\vec{v}_1\cdot\vec{v}_2 - \frac{(\vec{v}_1\cdot\vec{r})(\vec{v}_2\cdot\vec{r})}{r^2}\Big) - \frac{G^2 m_1 m_2(m_1+m_2)}{2r^2}\Big]$$

This Lagrangian was obtained by Einstein, Infeld & Hoffmann in the early days of GR. From this one can find e.g. the precession of Mercury's perihelion : $\varepsilon = \frac{24\pi^2 a^2}{T^2 (1-e^2)}$ in radians/revolution.

IV Dissipative dynamics

We now move towards the dynamics involving the radiation field $\bar{h}$.
In the $\bar{h}$ propagator, the dependence on v is still not homogeneous since: $e^{i\vec{k}\cdot\vec{r}} = \sum_{n\geq 0} \frac{(i\vec{k}\cdot\vec{r})^n}{n!}$ and $|\vec{k}| \sim v/r$.
We should expand in powers of $\vec{k}\cdot\vec{r}$.
This corresponds to a multipole expansion around the center-of-mass of the binary system.

The physical interpretation of such an expansion is quite clear: since we have integrated out modes with $k \sim 1/r$, we do not resolve the orbital scale any more, and treat the binary system as a single point-particle.

This effective point particle is coupled to $\bar h$. Since it is on-shell, we choose to impose the <u>transverse-traceless</u> gauge:

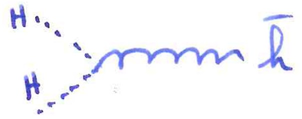
$$\boxed{\bar h_{0i} = 0 \qquad k^i \bar h_{ij} = 0 \qquad \bar h_{ij}\eta^{ij} = 0}$$

To lowest order in the multipole expansion, the $\bar h$ coupling (from the point-particle action) reduces to:

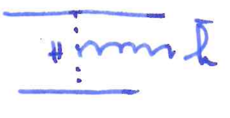
$$\boxed{\frac{1}{2m_P}\int dt\, \bar h_{ij}(x_{cm}, t) \sum_{A=1,2} m_A v_A^i v_A^j}$$

However we should not forget that $\bar h_{ij}$ also couples to the orbital modes H by the Einstein-Hilbert term!

$$\int d^4x\, \partial^2 H^2 \bar h$$



I give you directly the value of this diagram:



$$= \frac{i}{2m_P}\int dt\, \bar h_{ij}(x_{cm}, t) \sum_{A=1,2} m_A x_A^i \frac{d^2}{dt^2} x_A^j$$

Inverting the time derivative (by IBP), we finally find the coupling of $\bar h_{ij}$ once all potential modes H have been integrated out:

$$-\frac{1}{2m_P}\int dt\, \partial_t \bar h_{ij} \sum_A m_A x_A^i v_A^j = \boxed{-\frac{1}{4m_P}\int dt\, \partial_t^2 \bar h_{ij} \sum_A m_A x_A^i x_A^j}$$

Since $\bar h_{ij}$ is traceless, we might as well replace the $x_A^i x_A^j$ term with its traceless counterpart:

$-\frac{1}{4 m_P} \int dt\, \partial_t^2 \bar{h}_{ij}\, Q^{ij}$, $\boxed{Q^{ij} = \sum_A m_A \left(x_A^i x_A^j - \frac{1}{3} x_A^2 \delta^{ij} \right)}$

traceless quadrupole moment

You can show that in the TT gauge: $E_{ij} = \frac{1}{2 m_P} \partial_t^2 h_{ij}$

electric-type Riemann defined p. 4.

So the result that we get is perfectly consistent with an EFT reasoning: the coupling of a point-particle with quadrupole moment to gravity is $Q^{ij} E_{ij}$

From this coupling we can compute the imaginary part of the effective action. For this we use:

$$\frac{1}{k^2 - i\varepsilon} = PV \frac{1}{k^2} + i\pi\, \delta(k^2).$$

Integrating out the radiation field gives rise to the following Feynman diagram:

$$= \underbrace{\tfrac{1}{2}}_{\text{symmetry factor}} \frac{(-1)\, 1}{4} \int dt_1\, dt_2\, Q^{ij}(t_1)\, Q^{kl}(t_2) \langle E_{ij}(t_1) E_{kl}(t_2) \rangle$$

double line to denote that the binary is treated as a point-particle

<u>Exercise</u> (this one is not easy!): By using the imaginary part of the propagator, show that:

$$\mathrm{Im}\, S_{eff} = \frac{1}{20 m_P^2} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2|k|} k^4 \left| \underbrace{\tilde{Q}_{ij}(|k|)}_{\text{Fourier transform}} \right|^2$$

Hint: Use the following expression for E_{ij} (to lowest order):

$$E_{ij} = \frac{1}{2 m_P} \left(\partial_i \dot{\bar{h}}_{0i} + \partial_j \dot{\bar{h}}_{0i} - \ddot{\bar{h}}_{ij} - \partial_i \partial_j \bar{h}_{00} \right)$$

This time we are <u>not</u> allowed to use the transverse-traceless gauge for $\bar{h}$ since it is not an external field but enters in a propagator!

Now remind that: $\mathrm{Im}\, S_{eff} = \frac{T}{2}\int dE\, d\Omega\, \frac{d^2\Gamma}{dE\, d\Omega}$ & $P = \int E \frac{d^2\Gamma}{dE\, d\Omega}\, dE\, d\Omega$

$$\Rightarrow P = \frac{G}{5\pi T}\int_0^\infty dk\, k^6\, |\tilde{Q}_{ij}(k)|^2 = \frac{1}{10\pi T}\int_{-\infty}^{\infty} dk\, k^6\, |\tilde{Q}_{ij}(k)|^2$$

On the other hand, we can introduce the mean value over time:

$$\langle \dddot{Q}_{ij}(t)\, \dddot{Q}^{ij}(t)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} dt\, \dddot{Q}_{ij}(t)\, \dddot{Q}_{ij}(t)$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} dt\, \frac{dk}{2\pi}\frac{dk'}{2\pi}\, e^{i(k-k')t}\, k^3 k'^3\, \tilde{Q}_{ij}(k)\, \tilde{Q}_{ij}(-k')$$

$\int dt\, e^{i(k-k')t} = 2\pi\,\delta(k-k')$

$$= \frac{1}{2\pi T}\int_{-\infty}^{\infty} dk\, k^6\, |\tilde{Q}_{ij}(k)|^2$$

$$\Rightarrow \boxed{P = \frac{G}{5}\langle \dddot{Q}_{ij}\, \dddot{Q}_{ij}\rangle} \quad \text{quadrupole formula}$$

Conclusion: How to distinguish a BH from a NS

Up to now our computations are valid for both BH & NS modeled as point-particles. What distinguish them?

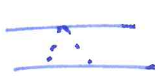
→ A: the finite-size effect!

E.g. $O_3 = \underset{\substack{\uparrow \\ \text{would be} \neq \text{for BH or NS.}}}{C_E}\int d\tau\, E_{\mu\nu} E^{\mu\nu}$

At which order would this show up in the perturbative expansion?

For a potential graviton: $E_{ij} \sim \frac{\partial^2 H}{m_p}$

$$\Rightarrow O_3 \sim \frac{C_E}{m_p^2}\,\frac{r}{v}\,\frac{1}{r^4}\,\frac{v}{r^2} \sim \frac{C_E}{m_p^2\, r^5}$$

On the other hand $C_E\int d\tau\, E_{ij}^2$ is a 2-graviton vertex , so it should scale as v^n to give a diagram  proportional to L.

$$\Rightarrow C_E \sim m_p^2\, \underbrace{(Gm)^5}_{\text{finite-size scale.}} \quad \text{and} \quad \boxed{O_3 \sim v^{10}}$$

So ϑ_3 is a $v^{10} \Leftrightarrow 5PN$ effect!

$\Rightarrow$ You have to perform computations up to this very high order if you want to discriminate between BH & NS in a GW signal.