Breakdown of Thermalization in Disordered Quantum Systems: Many Body Localization and its Consequences

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Contents

1 Introduction

The aim of this internship is to investigate further some properties of out-of-equilibrium quantum systems, the so-called Many-Body Localization (MBL). These systems exhibit behaviors indicating that they will never thermalize, as opposed to conventional quantum systems coupled to a bath which will progressively lose their coherence through this coupling and approach at long times the Gibbs-Boltzmann statistical equilibrium given by the equilibrium density matrix :

$$\rho_{eq}(H) = \frac{e^{-\beta H}}{Z}, \quad Z = Tr\left(e^{-\beta H}\right) \tag{1}$$

A set of general prescriptions have been formulated for isolated quantum system to reach their equilibrium distribution at long time : they go under the name of Eigenstate Thermalization Hypothesis (ETH), see e.g Nandkishore and Huse (2015) for a review (the basic idea is that the system will act as a reservoir for each of its subsytem). In this internship we will be interested about MBL system which violate this ETH.

MBL systems are the many-body analogue of Anderson localized sytems. Anderson localization was first discovered by studying electrons in a random potential. Anderson showed (Anderson (1958)) that in 1D and 2D systems, the wavefunction of the electrons is localized (i.e spread over a few sites only), whereas in 3D there is a mobility edge (which depends on the disorder strenght) that separates localized and extended states. This idea of localization is very important since a localized system will somehow never forget about its initial condition and so will never thermalize. Whereas Anderson localized systems have been observed in a lot of experiments, the discovery of MBL in an experimental setup is rather recent (Schreiber et al. (2015))

This distinction between localization and thermalization is only dynamical, i.e it will not show up in thermodynamic quantities since we expect the ergodic hypothesis to break down. We will therefore study time variation of quantities such as the magnetization in an Ising chain. Moreover, quenched disorder is essential to the apparition of localization.

After these general considerations, let's now become more precise about the exact aim of this internship, which settles in this general background but focuses on new developments on the subject. We will study the new concept of 'Time Crystals'. The basic idea is to try to obtain the temporal analogue of a spatial crystal, i.e a system for which continuous time translations are broken by a discrete one. This was first formulated in 2012 by Wilczek (Wilczek (2012)) and gave rise to some controversy about the existence of such a system at thermal equilibrium. Watanabe and Oshikawa (2015) then showed that it was indeed impossible to obtain such a system at thermal equilibrium, but the possibility is open in the framework of out-of-equilibrium quantum systems discussed above.

In particular, periodically driven systems could exhibit this behavior. In this case, if a system were driven with period T, any time crystal would break the discrete time-translational symmetry of the drive and only return to its initial state after discrete multiples of a time nT, where n is an integer. Furthermore, for this discrete time crystal to persist, it must be stabilized against heating to infinite temperature (a condition in which there is no time dependence and therefore no time-translation symmetry breaking) by imposing many-body localization, wherein localization prevents energy absorption from the drive (however some recent research point towards the existence of Floquet Time Crystals even without quenched disorder, see Huang et al. (2017))

With the stage set, Else et al. (2016) studied a theoretical model of a quantum system that would permit such time-crystal behavior. The system is a 1D chain of quantum spins subjected to an alternating drive that first flips all the spins and then allows the spins to interact with each other. The interactions take place in the presence of random, site-dependent external magnetic fields, which support the many-body localization phenomenon. Very recently, Zhang et al. (2017) observed the first experimental evidence of a discrete time crystal.

In this internship, we would like to study if the emergence of this Floquet Time Crystal is possible in a somewhat different model than the preceding ones. Indeed, most of the existing work has focused on very particular models where the time dependance is a square signal that flips all the spins (i.e $|\uparrow\rangle \rightarrow |\downarrow\rangle$) during one period and then flip them again during the second period, which explains the emergence of a period of $2T$ in quantities such as the magnetization (of course the whole procedure is not so simple, because there must be rigidity of the new period to external perturbation, which is not realized in a simple spin flip. See e.g Huang et al. (2017) for more details).

Consequently we will study an Ising chain periodically driven by a sinusoidal drive and look for the behavior of the magnetization under this drive. We will be particularly interested in the possible emergence of periodicities other than the one of the drive in the long-time limit.

This internship report is divided as follows : in Section 2 we will develop a general Floquet formalism for driven systems. In Section 3 we will present the particular model under study and how to solve it. In Section 4 we will discuss general features needed for the appearance of a Floquet Time Crystal, and finally in Section 5 we will present our numerical results.

2 Floquet theory

The general aim of this part is to explain how we can study the time evolution of an observable $\hat{O}$ in a periodically driven sytem. In particular, we are interested in observables that would have a periodicity which is different from the drive, even at long times. We will se that this is related to the eigenvalue distribution of the Hamiltonian.

To study the evolution of $\hat{O}$ we use Floquet theory for this system governed by a periodic Hamiltonian $\hat{H}(t+T) = \hat{H}(t)$ with $T = \frac{2\pi}{\Omega}$. This is the equivalent of Bloch theorem but adapted to time translations : it states that one can find a complete basis of solutions of the Schrödinger equation $|\Psi_\alpha(t)\rangle = e^{-i\mu_\alpha t}|\Phi_\alpha(t)\rangle$ where the Floquet modes $|\Phi_\alpha(t)\rangle$ are periodic : $|\Phi_\alpha(t+\tau)\rangle = |\Phi_\alpha(t)\rangle$, and μ_α are the (real) quasi-energies defined on the interval $\left[-\frac{\Omega}{2}, \frac{\Omega}{2}\right]$ (the first Brillouin zone). Expanding the evolved state $|\Psi(t)\rangle = \sum_\alpha R_\alpha e^{-i\mu_\alpha t}|\Phi_\alpha(t)\rangle$ on the Floquet basis, where $R_\alpha \equiv \langle\Phi_\alpha(0)|\Psi(0)\rangle$, we find :

$$\begin{aligned} O(t) &\equiv \langle\Psi(t)|\hat{O}|\Psi(t)\rangle \\ &= \sum_\alpha |R_\alpha|^2 O_{\alpha\alpha}(t) + \int_{-\infty}^{+\infty} F_O(\omega) e^{-i\omega t} d\omega \end{aligned} \tag{2}$$

where $F_O(\omega) = \sum_{\alpha\neq\beta} O_{\alpha\beta}(t) R_\alpha^\star R_\beta \delta(\omega - \mu_\alpha + \mu_\beta)$ and $O_{\alpha\beta}(t) = \langle\Phi_\alpha(t)|\hat{O}|\Phi_\beta(t)\rangle$. The first term (diagonal term) in eq (2) is clearly periodic of period T (or possibly smaller than T), so a change of periodicity like the one we are searching for can only be given by the second term. We now see that this depends on the eigenvalue distribution μ_α : if this distribution is a smooth continuum, then $F_O(\omega)$ will be smooth and the integral will average to zero at long times (Riemann-Lebesgue lemma). This is the case wich has been studied for example in Russomanno et al. (2012).

On the contrary, if the system has an important *pure-point* spectrum, F_O will be contituted of only a few delta-functions and new periodicities can arise. In this work we will manly be interested about these new periodicities, their relation with the Floquet spectrum and with the idea of Floquet Time Crystals.

In order to find the Floquet spectrum, one has to diagonalize the evolution operator over one period which reads :

$$\hat{U}(T) = \sum_\alpha e^{-i\mu_\alpha T}|\Phi_\alpha(0)\rangle\langle\Phi_\alpha(0)| \tag{3}$$

(to see this, one can check that the operator $\hat{U}(t) = \sum_\alpha e^{-i\mu_\alpha t}|\Phi_\alpha(t)\rangle\langle\Phi_\alpha(0)|$ satisfies the evolution equation)

3 Model : Periodically driven spin chain

3.1 Exact diagonalization

Let's now discuss the model under study. We consider a periodically driven spin chain whose Hamiltonian is defined by

$$\hat{H}(t) = -\sum_{j=1}^{L} J_j \left((1+\gamma)\sigma_j^x\sigma_{j+1}^x + (1-\gamma)\sigma_j^y\sigma_{j+1}^y\right) - \sum_{j=1}^{L} h_j(t)\sigma_j^z \tag{4}$$

In this equation $\sigma_j^{x,y,z}$ are the Pauli matrices at site j, γ is an anisotropy coefficient, J_j are random couplings, $h_j(t) = h_j^0 + h_1\cos(\Omega t)$ where h_j^0 are random fields chosen uniformly with mean h_m and width h_0, and $\Omega = \frac{2\pi}{T}$ is the frequency of the drive. As for the boundary conditions, we choose to take periodic boundary conditions $\sigma_{L+1} = \sigma_1$, although we have numerically checked that the boundary conditions do not change the behavior of the system.

If $\gamma = 0$, this is the XY model with a periodic transverse field. If $\gamma = 1$ (which is the case we will mainly consider in the numerical treatment), this is the transverse field Ising model (TFIM). This TFIM represents a paradigm solvable example of a quantum phase transition and it has been therefore the subject of a large literature (see Sachdev (2001)).

More specifically, if we consider only a constant non-disordered magnetic field $h_j(t) = \Gamma$ with a system in equilibrium at zero temperature and in the thermodynamic limit, the TFIM exhibits ferromagnetic ($\Gamma < J$) and paramagnetic ($\Gamma > J$) phases, separated by a quantum critical point at $\Gamma_c = J$. For $\Gamma < J$ and $L \to \infty$ there are two degenerate ground states related by a Z_2 symmetry. Spontaneous symmetry breaking selects unique ground state, in which spins align along the x-direction. On the other hand, for fields $\Gamma > J$ the ground state is non-degenerate and as the magnetic field Γ is increased, spins align more and more along the z-direction. The order parameter for the quantum phase transition is the ground state expectation value $\langle \sigma_i^x \rangle$. Furthermore we will refer to the ferromagnetic phase as the ordered phase, and to the paramagnetic one as the disordered phase.

We want now to get an evolution equation for the system under study. The dynamics can be obtained exactly using a Jordan-Wigner transformation and a time-dependent Bogolubov transformation (Lieb et al. (1961)), as we are going to show in the next sections.

The idea of mapping quantum spin 1/2 into (spinless) fermionic degrees of freedom has a long history in quantum mechanics. It can be seen as a very natural one from the point of view of local Hilbert space, since the two up/down states (eigenstates of σ^z) can be described in terms of presence/absence of a fermionic particle (eigenstates of the local charge $n = c^\dagger c$) and analogously the spin flip operators $\sigma^\pm$ as creation/annihilation of such a particle, i.e. $\sigma^+ = f^\dagger$, $\sigma^- = f$ and $\sigma^z = 1 - 2n$. The problem with this naive identification comes from the statistics, namely from the fact that spins on different sites are expected to commute, while fermions anticommute :

$$\left\{ c_i, c_j^\dagger \right\} = \delta_{ij} \tag{5}$$

In one dimension there is a way to overcome such a difficulty, namely to have a fermionic representation of the spin which is consistent with the statistics, through the so called Jordan-Wigner transformation. The idea is to introduce a fermionic representation of the quantum spin at each lattice site j of the form

$$\sigma_j^x = K_j (c_j^\dagger + c_j) \tag{6}$$

$$\sigma_j^y = i K_j (c_j^\dagger - c_j) \tag{7}$$

$$\sigma_j^z = (2 c_j^\dagger c_j - 1) \tag{8}$$

where the string operator K_j sets the proper sign by counting for the number of fermions preceding the j-th site :

$$K_j = \prod_{i<j} (1 - 2 c_i^\dagger c_i) = \exp\left(i\pi \sum_{i<j} c_i^\dagger c_i \right) \tag{9}$$

Using the fermionic algebra one finds the following expression of $\hat{H}$:

$$\hat{H}(t) = -2 \sum_{i=1}^{L} J_i (\gamma c_i^\dagger c_{i+1}^\dagger + c_i^\dagger c_{i+1} + h.c) - 2 \sum_{i=1}^{L} h_i(t) c_i^\dagger c_i + \sum_{i=1}^{L} h_i(t) \tag{10}$$

In this equation we have set the following boudary condition : $c_{L+1} = (-1)^{N_F+1} c_1$ where $N_F = \sum_{i=1}^{L} c_i^\dagger c_i$ is the total number of fermions whose parity is conserved by the Hamiltonian. In the rest of the paper we will assume that N_F is even, which is justified since we start from the ground state of the hamiltonian and consider averages of operators which couple only an even number of fermions.

Following Young and Rieger (1996) and recasting the c_i's in a vector $\psi = (c_1, ..., c_L, c_1^\dagger, ..., c_L^\dagger)^T$, we can write the Hamiltonian as :

$$\hat{H}(t) = \sum_{ij=1}^{2L} \psi_i^\dagger \tilde{H}_{ij}(t) \psi_j \tag{11}$$

where $\tilde{H}$ is a $2L \times 2L$ matrix which has the following form :

$$\begin{pmatrix} A & B \\ -B & -A \end{pmatrix} \tag{12}$$

where

$$A(t) = \begin{pmatrix} -h_1(t) & -J_1 & 0 & \dots & (-1)^{N_F} J_L \\ -J_1 & -h_2(t) & -J_2 & & \\ 0 & \ddots & \ddots & \ddots & \\ & & -J_{L-2} & -h_{L-1}(t) & -J_{L-1} \\ (-1)^{N_F} J_L & & 0 & -J_{L-1} & -h_L(t) \end{pmatrix} \in S_L(\mathbb{R}) \tag{13}$$

and

$$B = \gamma \begin{pmatrix} 0 & -J_1 & 0 & \dots & (-1)^{N_F+1} J_L \\ J_1 & 0 & -J_2 & & \\ 0 & \ddots & \ddots & \ddots & \\ & & J_{L-2} & 0 & -J_{L-1} \\ (-1)^{N_F} J_L & & 0 & J_{L-1} & 0 \end{pmatrix} \in A_L(\mathbb{R}) \tag{14}$$

Be careful that $\tilde{H}$ is not the Hamiltonian of the system, just a matrix in the basis of the ψ_i's. It is called the Bogoliubov-de Gennes Hamiltonian, by analogy with the BCS theory of superconductivity.

We want now to diagonalize this hamiltonian. It is easy to see that if (u, v) is an eigenvector of $\tilde{H}$ with eigenvalue ε, then (v, u) is an eigenvector associated to the eigenvalue $-\varepsilon$. $\tilde{H}$ is a symmetric matrix so we can diagonalize it in an orthogonal basis at $t = 0$:

$$\tilde{H}(0) = ODO^T$$

with

$$D = diag(\varepsilon_1, \dots, \varepsilon_L, -\varepsilon_1, \dots, -\varepsilon_L) \tag{15}$$

The eigenvector $(u_\mu^0, v_\mu^0) \in \mathbb{R}^{2L}$ associated to the eigenvalue $\varepsilon_\mu \geq 0$ verifies the following equations :

$$\begin{aligned} A(0)u_\mu^0 + Bv_\mu^0 &= \varepsilon_\mu u_\mu \\ -Bu_\mu^0 - A(0)v_\mu^0 &= \varepsilon_\mu v_\mu \end{aligned} \tag{16}$$

plus the orthogonality conditions

$$\begin{aligned} \|u_\mu^0\|^2 + \|v_\mu^0\|^2 &= 1 \\ u_\mu^0 \cdot u_\nu^0 + v_\mu^0 \cdot v_\nu^0 &= 0 \quad (\mu \neq \nu) \\ u_\mu^0 \cdot v_\nu^0 &= 0 \quad (\mu \neq \nu) \end{aligned} \tag{17}$$

We can now define new fermionic operators (that are just the c_i's in a rotated basis) :

$$\begin{pmatrix} \gamma_\mu \\ \gamma_\mu^\dagger \end{pmatrix} = O^T \begin{pmatrix} c_i \\ c_i^\dagger \end{pmatrix} \tag{18}$$

which is equivalent to

$$\begin{aligned} \gamma_\mu &= \sum_{i=1}^{L} u^0_{\mu,i} c_i + v^0_{\mu,i} c^\dagger_i \\ c_i &= \sum_{\mu=1}^{L} u^0_{\mu,i} \gamma_\mu + v^0_{\mu,i} \gamma^\dagger_\mu \end{aligned} \tag{19}$$

In this basis the Hamiltonian taks the simple form

$$\hat{H}(0) = \sum_\mu \varepsilon_\mu \gamma^\dagger_\mu \gamma_\mu - \varepsilon_\mu \gamma_\mu \gamma^\dagger_\mu = 2 \sum_\mu \varepsilon_\mu \left(\gamma^\dagger_\mu \gamma_\mu - 1/2 \right) \tag{20}$$

We want now to find the temporal evolution of the (Heisenberg picture) operators $c_i(t)$. To this aim we will use the evolution equation for c_i :

$$i \frac{dc_i}{dt} = \left[c_i, \hat{H} \right] \tag{21}$$

The calculation of this commutator using fermionic rules is straightforward and we find :

$$\left[c_k, \hat{H} \right] = 2 \sum_j A_{kj} c_j + B_{kj} c^\dagger_j \tag{22}$$

We now make the following hypothesis :

$$c_k(t) = \sum_\mu u_{\mu k}(t) \gamma_\mu + v^\star_{\mu k}(t) \gamma^\dagger_\mu \tag{23}$$

where $(u_\mu, v_\mu) \in \mathbb{C}^{2L}$ are complex time-dependent coefficients with initial condition $u_\mu(t=0) = u^0_\mu$ and $v_\mu(t=0) = v^0_\mu$, and the γ_μ are the operators that diagonalize the initial hamiltonian. One can check that this form of c_k satisfies the evolution equation provided the coefficients (u_μ, v_μ) satisfy the following differential equation :

$$i \frac{d}{dt} \begin{pmatrix} u_\mu \\ v_\mu \end{pmatrix} = 2\tilde{H}(t) \begin{pmatrix} u_\mu \\ v_\mu \end{pmatrix} \tag{24}$$

(and similarly $\left(v^\star_\mu, u^\star_\mu \right)^T$ satisfies the same equation but with an initial condition $\left(v^0_\mu, u^0_\mu \right)^T$ corresponding to a negative eigenvalue of $\tilde{H}(0)$)

It is now easy to find the temporal evolution of quantities such as the magnetization. Indeed, if one starts from the ground state $|\Psi_0\rangle$ of $\hat{H}(t=0)$ i.e $\gamma_\mu |\Psi_0\rangle = 0$, then one finds the following simple expression for the (quantum) average of the magnetization $\hat{m} = \frac{1}{L} \sum_j \sigma^z_j = \frac{1}{L} \sum_j \left(2c^\dagger_j c_j - 1 \right)$:

$$m(t) = \frac{2}{L} \sum_\mu \| v_\mu(t) \|^2 - 1 \tag{25}$$

Similarly, we will need the x time correlation function $\langle \sigma^x_1(t) \sigma^x_1(0) \rangle$ (computed on the first site otherwise it would involve the string operator defined in Equation 9, which is complicated). It is given by the expression :

$$\langle \sigma^x_1(t) \sigma^x_1(0) \rangle = \sum_\mu (u_{\mu 1}(t) + v_{\mu 1}(t))(u^0_{\mu 1} + v^0_{\mu 1}) \tag{26}$$

3.2 Floquet Spectrum

Let's now apply Floquet theory to the evolution equation (24). To find the Floquet spectrum $\{\mu_\alpha\}$, one has to diagonalize the Floquet operator which is the evolution operator over a period T of the drive, see equation 3.

Let's adopt a quantum mechanical notation and denote $|e_\mu(t)\rangle = (u_\mu(t), v_\mu(t))^T$ [1], $\mu = 1, ..., L$ the evolved positive eigenvector of $\tilde{H}(0)$ (corresponding to a positive eigenvalue), and similarly $|e_\mu(t)\rangle = \left(v^\star_\mu(t), u^\star_\mu(t)\right)^T$, $\mu = -1, ..., -L$ the evolved negative eigenvector of $\tilde{H}(0)$ (corresponding to a negative eigenvalue). Then the evolution operator is defined by :

$$|e_\mu(t)\rangle = \hat{U}(t)\,|e_\mu(0)\rangle \tag{27}$$

So in the orthogonal basis $\{e_\mu(0)\}$ the evolution operator has the following expression :

$$\hat{U}(t) = \sum_{\mu\nu} \langle e_\nu(0)|e_\mu(t)\rangle\, |e_\nu(0)\rangle\, \langle e_\mu(0)| \tag{28}$$

where μ and ν run over positive and negative values. Once again one can check that, due to the symmetry of the sytem, the Floquet eigenvalues come in pairs $(\mu_\alpha, -\mu_\alpha)$, so we will restrict the Floquet spectrum in the interval $[0, \frac{\pi}{T}]$.

As done in Russomanno et al. (2012), it is easier numerically to diagonalize the matrix

$$\mathcal{A} = -i\left(1 - \hat{U}(\tau)\right)\left(1 + \hat{U}(\tau)\right)^{-1} \tag{29}$$

because the eigenvalues algorithm finds first the largest modulus eigenvalue, which is ill-defined for a unitary matrix whose eigenvalues are of modulus 1. The eigenvalues of $\mathcal{A}$, denoted by a_α, are related to the Floquet spetrum by $\mu_\alpha = \frac{\omega}{\pi}\arctan(a_\alpha)$.

4 Floquet Time Crystal criterion

We would now like to study more in detail the Time Crystal properties of this system. As we have said in the introduction, the criterion of period doubling usually invoked in the recent studies to identify a Time Crystal is not adapted to our case, since we did not choose to use a 'spin-flip' hamiltonian. We will rather look at the emergence of other periodicities in the magnetization. To this aim we can look at the Fourier transform of the magnetization or at the Floquet spectrum.

Indeed, if the Floquet spectrum $\{\mu_\alpha\}$ is peaked around a value $\bar{\mu}$ (and consequently also peaked around $-\bar{\mu}$ by the remark following equation (28)), then we should see a new periodicity of $2\bar{\mu}$ appearing (we recall that $F_O(\omega) = \sum_{\alpha\neq\beta} O_{\alpha\beta}(t) R^\star_\alpha R_\beta \delta\left(\omega - \mu_\alpha + \mu_\beta\right)$ gives the non-diagonal average of the mean value of an observable, by which new periodicities can arise). However, this non-diagonal average involves also matrix elements of O and initial conditions, so there is not a direct and evident relation between the Floquet spectrum and the new periodicities appearing in the time evolution of $\langle O(t)\rangle$. This is why we will always examine the Floquet spectrum on the one hand and the time evolution of the magnetization on the other hand in order to be certain to capture all the features in our model.

We have in fact observed such a peaked Floquet spectrum in some cases, but we will argue below that it is not related to a Time Crystal. Indeed, as formulated by e.g Huang et al. (2017), the emergence of a Time Crystal is subject to three conditions :

1. Time-translation symmetry breaking : there should exist an observable whose average is not invariant by translation of τ (period of the Hamiltonian)

[1] One can check that the evolution equation 24 preserve the orthogonality conditions 17, i.e that the evolved states $|e_\mu(t)\rangle$ form an othogonal basis

2. Rigidity: this observable shows a fixed oscillation frequency without fine-tuned Hamiltonian parameters

3. Persistence: the non-trivial oscillation with fixed frequency must persist to indefinitely-long time when first taking system size L to the thermodynamic limit.

We will see in the following that we will not find a regime of parameters where these three conditions are enforced in the model under study.

5 Numerical Results

5.1 Benchmark

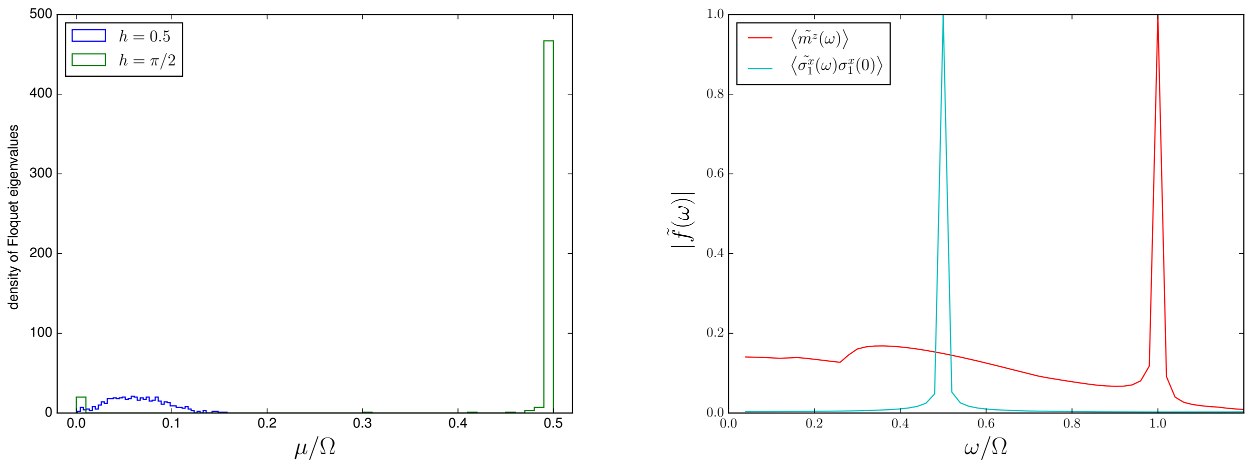
In order to be sure to be able to identify a Time Crystal with our observables, we have tried to implement a version of the binary Hamiltonian used by Khemani et al. (2016) and Else et al. (2016) to explore the Time Crystal phase, which is given by :

$$\begin{aligned}
\hat{H}(t) &= \begin{cases} \hat{H}_z & \text{if } 0 \leq t < T_1 \\ \hat{H}_x & \text{if } T_1 \leq t < T = T_1 + T_2 \end{cases} \\
\hat{H}_z &= -\sum_{i=1}^{L} h_i \sigma_i^z \\
\hat{H}_x &= -\sum_{i=1}^{L-1} J_i \sigma_i^x \sigma_{i+1}^x
\end{aligned} \tag{30}$$

where $J_i \in [0, J]$ and $h \in [0, h]$. Note that the boundary conditions are open, contrary to the Hamiltonian that we use. We will try to recover their results with another approach.

5.1.1 Clean case

If we choose a clean (non-disordered) field of value $hT_1 \approx \frac{\pi}{2}$, the evolution is approximately a flip of the spin along the x direction (because then $\exp\Big(ihT_1 \sum_{i=1}^{L} \sigma_i^z\Big) = \prod_{i=1}^{L} i\sigma_i^z$ flips the x spin eigenstates $\frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle)$) followed by an Ising interaction. The x magnetization will then be $2T$ periodic. Khemani et al. (2016) and Else et al. (2016) then argue for the stability of this phase to different perturbations of the parameters, including disorder and interactions (modeled by a $\sigma_i^z \sigma_{i+1}^z$ term). In this part we will not try to study the stability of the system to perturbations (necessary to prove the existence of a time crystal) and rather look at the simplest case of no disorder and interactions, trying to characterize the emergence of a time crystal.



(a) (b)

Figure 1: **(a)** Floquet spectrum for the drive described in equation 30. Parameters : $L = 500$, $\gamma = 1$, $J = 0.5$, $T_1 = 1$, $T_2 = 1$ **(b)** Fourier transform of the z magnetization (showing a peak at $\omega = \frac{2\pi}{T}$) and of the x time correlation function (showing a peak at $\omega = \frac{\pi}{T}$). Parameters : $L = 150$, $\gamma = 1$, $J = 0.5$, $h = \frac{\pi}{2}$, $T_1 = 1$, $T_2 = 1$. The heights of the peaks have been rescaled to compare them on the same figure.

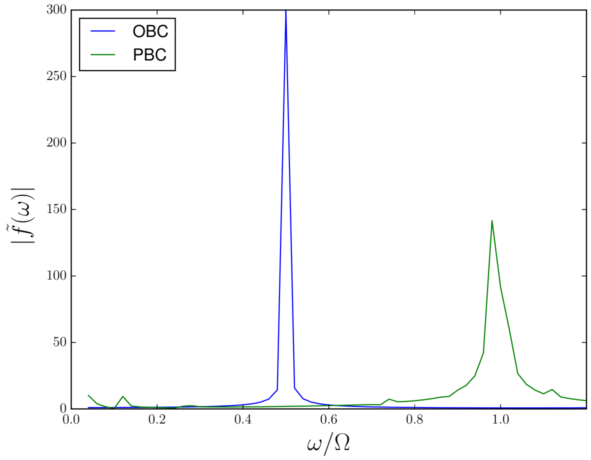
Figure 1 contains the Floquet spectrum for different values of the field h. We notice the emergence of a very peaked Floquet spectrum for $hT_1 \approx \frac{\pi}{2}$, whereas the spectrum is continuous for other values of h (although the peak can survive for a rather large interval of values of h, see Khemani et al. (2016) for further information). This peculiar feature of the Floquet spectrum enable us to identify the time crystal phase, but there is a subtelty in the argument. We notice that the peak of the Floquet spectrum is located around $\mu = \frac{\pi}{T}$ (and also around $\mu = -\frac{\pi}{T}$ by the symmetry of the system), and there is also a little peak at $\mu = 0$. Therefore, in order to observe oscillations of frequency $\frac{\pi}{T}$, we should select an observable which couples only the two peaks $\frac{\pi}{T}$ and 0. An observable which couples all Floquet modes, like the magnetization along the z axis, will trigger oscillations between the two main peaks with frequency $\omega = \frac{\pi}{T} - (-\frac{\pi}{T}) = \frac{2\pi}{T}$ and therefore will be undiscernable from the frequency of the drive, see Figure 1.

Indeed such a system is designed to observe a period doubling in the x magnetization only. Since the (quantum) average of the x magnetization in our model is 0 (because it involves the mean value of an odd number of fermionic operators), we chose to display its time correlation function $\langle \sigma_i^x(t)\sigma_i^x(0)\rangle$. In Figure 1 we show the evolution of the time correlation function of the x magnetization on the first site, $\langle \sigma_1^x(t)\sigma_1^x(0)\rangle$. It clearly shows oscillations of period $2T$ as expected.

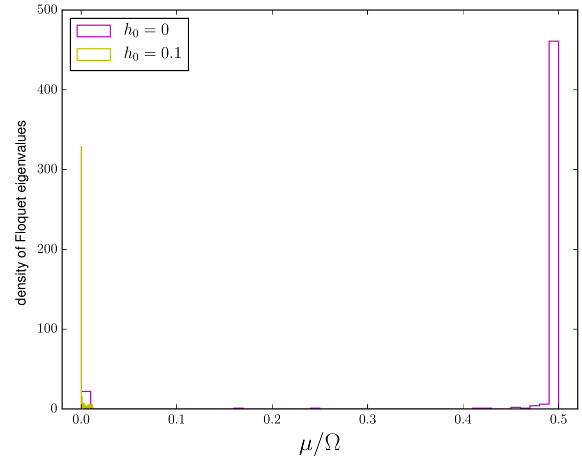
As a conclusion on this peculiar hamiltonian, we have clearly observed the emergence of a Floquet time crystal by looking at the Floquet spectrum and by showing oscillations of an observable designed to couple only some parts of the Floquet spectrum. Going back to our initial Hamiltonian, we can now begin the search for a time crystal by looking at the properties of the Floquet spectrum, and maybe try to design an observable which will have the right behavior (but we have not tried to push this idea of 'designed observable' further).

5.1.2 Changing boundary conditions and adding disorder

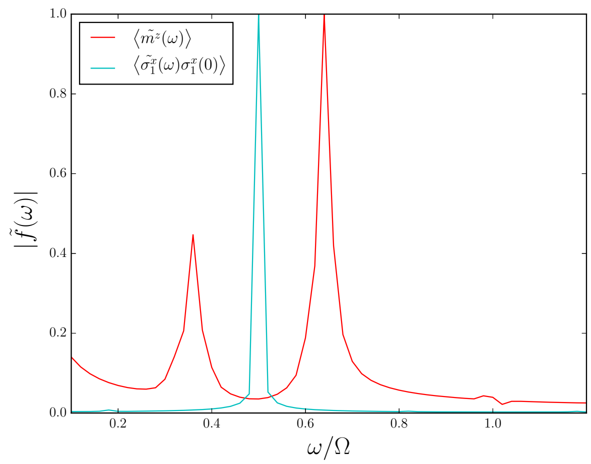
In this part we will explore the behavior of the system driven by the Hamiltonian 30 when changing the boundary conditions and adding disorder.



(a)



(b)



(c)

Figure 2: **(a)** Fourier transform of the x time correlation function for different boundary conditions. Parameters : $L = 100$, $\gamma = 1$, $J = 0.5$, $h = \frac{\pi}{2}$, $T_1 = 1$, $T_2 = 1$. **(b)** Floquet spectrum with and without disorder. Parameters : $L = 500$, $\gamma = 1$, $J = 0.5$, $T_1 = 1$, $T_2 = 1$, $h_m = \frac{\pi}{2}$. **(c)** Fourier transform of the z magnetization and the x time correlation function with disorder. $L = 100$, $\gamma = 1$, $J = 0.5$, $h_m = \frac{\pi}{2}$, $h_0 = 0.1$ $T_1 = 1$, $T_2 = 1$.

First, we have noted that the breaking of time-translational symmetry is very sensitive to the boundary conditions of the system : in Figure 2 **(a)** we observe that the $\frac{\pi}{T}$ peak in the x correlation function totally disappears if we put periodic boundary conditions (PBC) instead of open ones (OBC). Hence we deduce that the $\frac{\pi}{T}$ periodicity is due to edge modes. A careful study in Khemani et al. (2016) and Thakurathi et al. (2013) shows that this effect is due to edge Majorana fermions. We have verified that the different boundary conditions do not change the behavior of the system with our initial hamiltonian.

Then we have tried to add some disorder to this Hamiltonian, by replacing the constant magnetic field by a disordered one of mean h_m and width h_0. To our surprise we have observed a very different Floquet spectrum when the disorder is switched on, even at very low disorder (Figure 2 **(b)**). The behavior of the x time correlation function stays the same, but the z magnetization changes drastically (Figure 2 **(c)**). We do not have yet any explanation for this phenomenon, although this is not in contradiction with Khemani et al. (2016) since the x time correlation function seems unaffected by this change, hence the time crystal phase is stable.

This observation seems to invalidate the simple relation between the shape of the Floquet spectrum, but this is only an apparent contradiction : the non-diagonal term in the average of an observable given by $\sum_{\alpha\neq\beta} O_{\alpha\beta}(t) R_\alpha^\star R_\beta e^{i(\mu_\beta-\mu_\alpha)t}$ is a sum of functions with different periodicities, hence it can have any periodicity. As a conclusion on this section, we stress that the emergence of a Time Crystal is strongly related to the choice of an observable, and that there is *a priori* no obvious relation between the shape of the Floquet spectrum and the appearance of new periodicities, although in all cases with our initial Hamiltonian we have observed the usual relation between the two (see section 4). This is why we will always observe the entire time evolution of an observable, and not only the Floquet spectrum which is computationally much quicker to obtain (one has to do the evolution over one period of the drive only).

5.2 Harmonic driving

We now go to the new case of an harmonic drive with our initial Hamiltonian 4. In our numerical treatment, we suppose that the nearest-neighbor coupling is non-disordered, $J_i = J$ whereas the time-dependent magnetic field is a sum of a disordered piece and a sinusoidal excitation : $h_j(t) = h_j^0 + h_1 \cos(\Omega t)$ with h_j^0 random fields obtained from a uniform distribution on $[-h_0 + h_m, h_0 + h_m]$ (h_m and h_0 are respectively the mean and the width of the field)

We numerically diagonalize $\tilde{H}$ then implement equation (24) to find the time-evolved value of $|e_\mu(t)\rangle$ for $\mu = 1, ..., L$, which enable us to compute the mean value of any operator. Before going to the full disordered case, we have first examined the results obtained in the clean (non-disordered) case as a sanity check, as we will now explain.

5.2.1 Clean case

In this section we will consider only non-disordered fields, i.e $h_0 = 0$. This case has already been widely studied in the literature so we will use the known results as a check for the validity of our code.

The first check that one can think of is whether the magnetization given by eq 25 reproduces the known results in the case of a 'quantum quench' with uniform field when one can go to Fourier space to get an exact solution for m. We have indeed confirmed this result.

We have then examined the behavior of the magnetization when going to a driving sinusoidal field. We expected the magnetization to synchronize with the drive at long times. We observed in two different cases the emergence of new periodicities, that we will argue not to be related to a Floquet time crystal in the following.

To examine the periodicities we will plot the Fourier transform of the magnetization $|\tilde{m}(\omega)|$. It will always show a peak at $\omega = \frac{2\pi}{T}$, the frequency of the drive, and we will look for other peaks appearing.

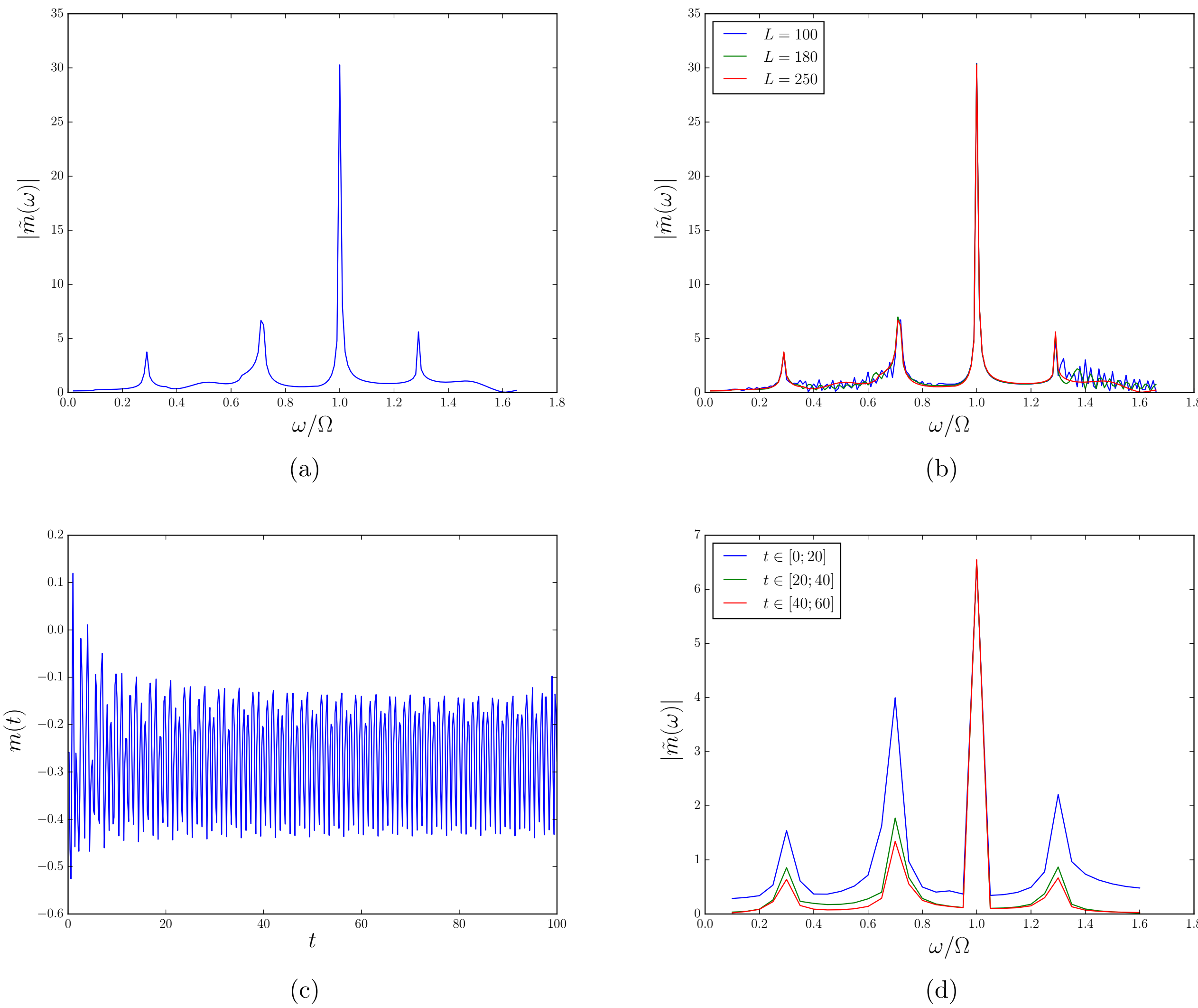
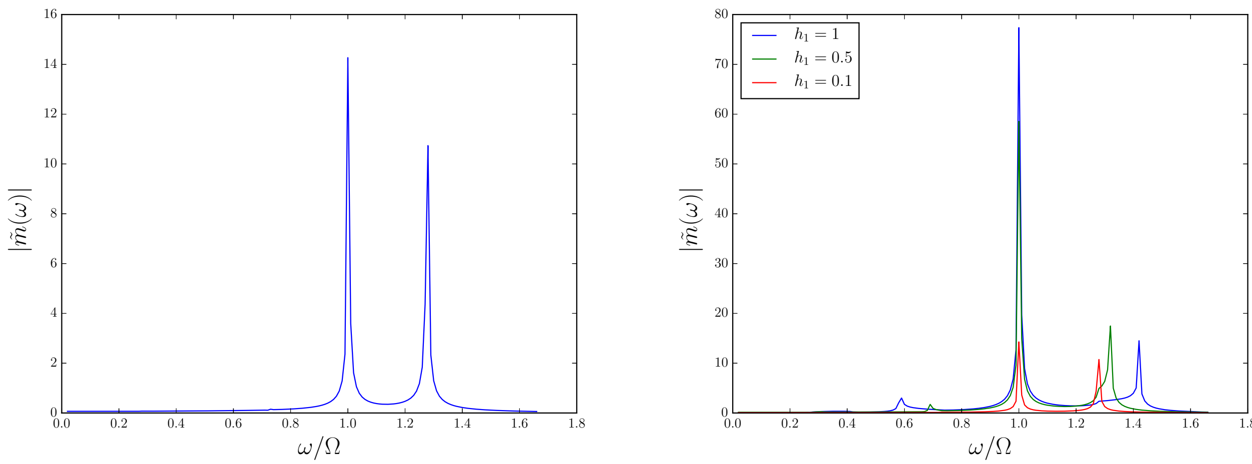


Figure 3: **(a)** Fourier transform of the magnetization with a clean periodic drive and parameters : $h_m = 1$, $h_0 = 0$, $h_1 = 1$, $J = 1$, $L = 250$, $T = 1$, $\gamma = 1$ and evaluated using a total time of $t = 100$. We observe small new periodicities appearing. **(b)** Same parameters and plot, but for different values of L. **(c)** Plot of the magnetization versus time with the same parameters. It shows a power-law decay of the secondary oscillations. **(d)** Fourier transform of the magnetization over different time intervals, keeping the other parameters fixed. The satellite peaks decay with time.

Synchronization with the drive : decaying oscillations As illustrated in Figure 3, we first observed the appearance of other periodicities for a broad range of parameters close to the ones indicated in the legend. We have checked that these satellite peaks do not decrease with the system size, but decay slowly to zero if we go to very large times. This result is in agreement with e.g Bhattacharyya et al. (2012) and Russomanno et al. (2012), which observed power-law decaying oscillations in a model similar to the one we used. Since it is clearly not the kind of behavior that we are after and that these oscillations needed rather large computational time in our setting, we did not investigated further in this direction.



(a) (b)

Figure 4: **(a)** Fourier transform of the z magnetization for parameters : $h_m = 0$, $h_0 = 0$, $h_1 = 0.1$, $L = 100$, $\gamma = 1$, $J = 1$, $T = 1$. There is a clear peak at $\omega_\star = 8J$. **(b)** Same figure and parameters but we change the drive strenght h_1. The secondary peak is less and less visible.

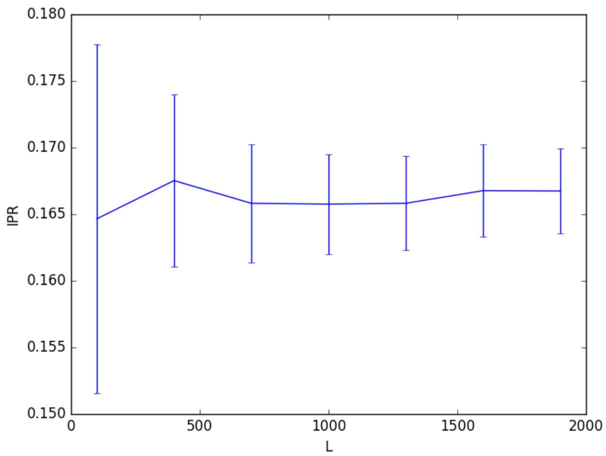
Zero time average field In the case of zero time average initial field and small driving field i.e $h_0 = 0$, $h_m = 0$ and h_1 small, we observed the emergence of a new periodicity surviving at long times and large sizes (see Figure 4). This very peculiar feature does not seem to exhibit decay at large times, but vanishes as soon as $h_m \neq 0$ or h_0, h_1 becoming large. We have found an explanation to this behavior, the detailed calculations are explained in Appendix A (since it is somehow technical, we chose not to include them in the main text). The main reason for the appearance of this other periodicity at $\omega_\star = 8J$ (see Appendix A) is that the ground state of the initial Hamiltonian with zero field is very peculiar : all spins up or all spins down (it is the ground state of the zero field and zero temperature Ising model). Since again this feature clearly does not respect the conditions for the emergence of a time crystal, we will not discuss this feature further.

5.2.2 Disordered case

Inverse Participation Ratio To begin with, we have first checked that the initial Hamiltonian has always localized states, by e.g plotting the Inverse Participation Ratio (IPR) of a state μ :

$$IPR(\mu) = \sum_j |u_{j\mu}|^4 + |v_{j\mu}|^4 \tag{31}$$

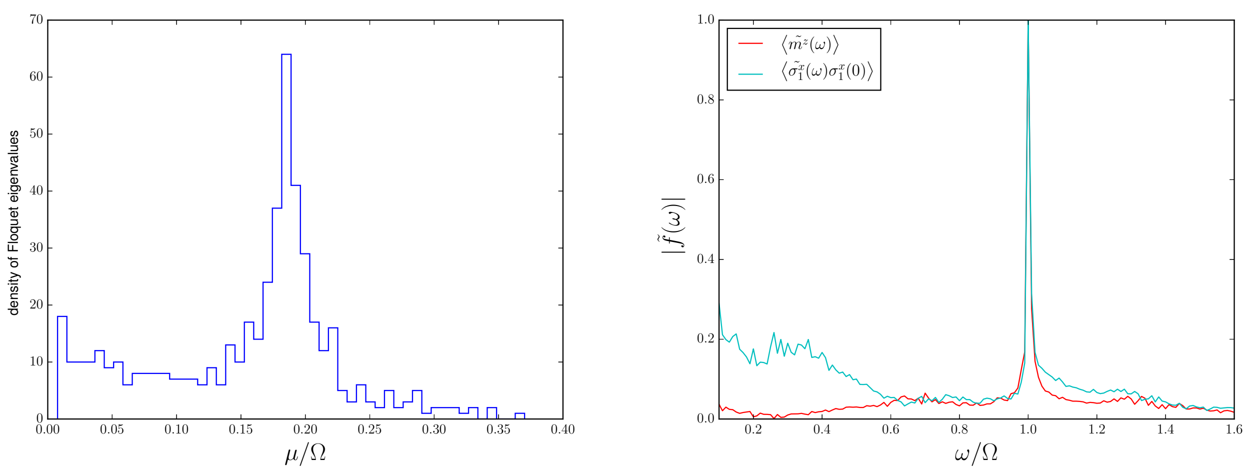
This IPR is roughly the inverse of the number of states $|e_\mu(t)\rangle$ is spread on. We find that this ratio does not depend on μ and does not decrease with L for all nonzero values of h_0, which indicates that the states are spread on very few states, i.e they are localized. On Figure 5 we show the quantity $IPR = \frac{1}{L}\sum_\mu IPR(\mu)$.



(a) (b)

Figure 5: **(a)** IPR plotted against different values of L for parameters $J = 1$, $h_0 = 1$, $\gamma = 1$. We average over 50 disorder realizations. The IPR is constant, signaling a localized state. This plot is the same for all nonzero values of h_0, even very low. **(b)** IPR plotted against different values of L for parameters $J = 1$, $h_0 = 0$ (no disorder), $\gamma = 1$. The IPR goes to zero at large sizes, signaling a delocalized state

Low frequency and comparison with a Time Crystal We have then tried to observe the behavior of the Floquet spectrum for certain regimes of parameters. This Floquet spectrum for a typical value of disorder and drive is represented in Figure 6 : the density of Floquet eigenvalues is smooth and distributed equally in the interval $[0, \frac{\pi}{T}]$, so there is no reason for a new periodicity to appear, and indeed we do not observe such a phenomenon in the Fourier transform of the magnetization, both in x and z.



(a) (b)

Figure 6: **(a)** Floquet spectrum density for parameters : $h_m = 1$, $h_0 = 1$, $h_1 = 1$, $L = 500$, $\gamma = 1$, $J = 1$, $T = 1$ and averaged over 50 realizations. **(b)** z magnetization and x time correlation function for parameters : $h_m = 1$, $h_0 = 1$, $h_1 = 1$, $L = 100$, $\gamma = 1$, $J = 1$, $T = 1$ and averaged over 20 realizations. There is no particular structure appearing other than the drive frequency. The heights of the peaks have been rescaled to compare them on the same figure.

For low drive frequencies we always observed a Floquet spectrum which was continuous and spread over the entire interval $[0, \frac{\pi}{T}]$. However, we observed some different behavior for high frequencies.

High frequency We have tried to go to a high drive frequency. In this case, since the Floquet eigenvalues are allowed to lie in a much larger interval $[0, \frac{\pi}{T}]$, it becomes clear that the width of the Floquet spectrum is mainly controlled by the disorder width h_0 (see Figure 7 **(a)**). For h_0 not too large, the Floquet spectrum is spread on few eigenvalues as compared to the entire interval $[0, \frac{\pi}{T}]$, and so we should see new periodicities appearing at low frequencies compared to the drive. This is indeed what we observe (Figure 7 **(b)**). Since the Floquet spectrum is still continuous, these secondary oscillations will as before wash away in the long-time limit du to destructive interferences between different Floquet frequencies.

For other regimes of parameters, we always observed a Floquet spectrum becoming continuous in the $L \to \infty$ limit. As a conclusion, we believe that in this setting we will not observe a Floquet time crystal appearance.

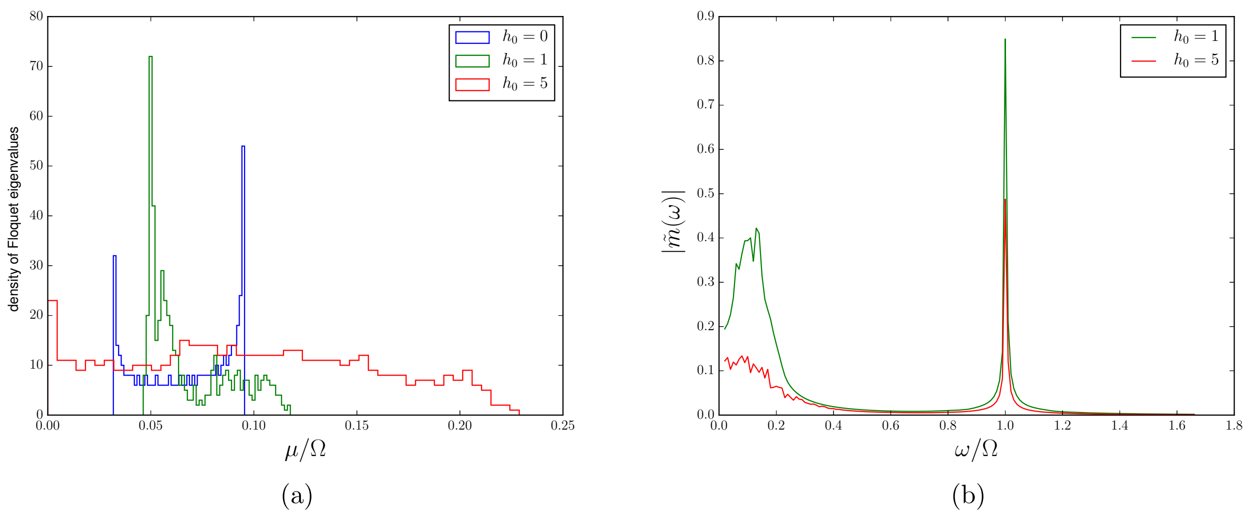


Figure 7: **(a)** Floquet spectrum density for parameters : $h_m = 1$, $h_1 = 1$, $L = 500$, $\gamma = 1$, $J = 1$, $T = 0.1$ for different values of h_0 and averaged over 50 realizations. **(b)** Fourier transform of the z magnetization for parameters : $h_m = 1$, $h_1 = 1$, $L = 100$, $\gamma = 1$, $J = 1$, $T = 0.1$, evaluated using a total time of $t = 10$, and with different values of h_0. The appearance of new periodicities around $\omega = 5 - 15$ is coherent with the Floquet spectrum localized around $\mu = 3 - 6$.

6 Conclusion

In this internship report, we have examined the possibility of an emergence of a Floquet Time Crystal in an harmonically driven spin chain, contrary to the majority of the literature which up to now has focused on simpler drives in systems designed to break the time-translation symmetry of T down to $2T$. After a comparison with a system with already known Time Crystal behavior, which enabled us to settle in what kind of observations this Time Crystal is visible, we have numerically implemented our system. In some cases we have found new periodicities appearing but we argue that they are not the type of periodicities characterizing a Time Crystal. This seems to indicate that there is no Time Crystal emergence in this harmonically driven system.

A Zero mean field

In this Appendix we will show that in the case of $h_m = 0$, $h_0 = 0$, $\gamma = 1$ and h_1 small, the Floquet spectrum is sharply peaked around $\mu = 4J$, and consequently there is a new periodicity of $\omega_\star = 8J$ appearing. In this special case the Hamiltonian takes the form $\hat{H}(t) = \hat{H}_0 + \hat{H}_1 \sin(\Omega t)$ with :

$$\hat{H}_0 = -J \sum_{j=1}^{L} \sigma_j^x \sigma_{j+1}^x, \quad \hat{H}_1 = -h_1 \sum_{j=1}^{L} \sigma_j^z \tag{32}$$

We start from the ground state of the initial hamiltonian $\hat{H}_0$ which is any linear combination (in the Hilbert space) of the states $|++\cdots+\rangle_x$ and $|--\cdots-\rangle_x$. In the language we used to diagonalize the Hamiltonian, this initial state corresponds to an eigenvector of the initial Bogoliubov-de Gennes Hamiltonian given by Equation 12, whereas the perturbation $\tilde{H}_1$ has the following simple form :

$$\begin{pmatrix} I_L & 0 \\ 0 & -I_L \end{pmatrix} \tag{33}$$

In the particular setting under study, the eigenvalues of the initial Hamiltonian $\tilde{H}_0$ are all equal to $\varepsilon = \pm 2J$, associated to the positive eigenvectors $\left|e_\mu^0\right\rangle = (u_\mu^0, v_\mu^0)^T$ (the negative ones being $(v_\mu^0, u_\mu^0)^T$).

We will now prove that at each instant t we have $\tilde{H}(t)\left|e_\mu^0\right\rangle \propto \left|e_\mu^0\right\rangle$ up to terms of order $h_1 \ll J$. Together with the evolution equation $i\frac{d}{dt}\left|e_\mu\right\rangle = 2\tilde{H}(t)\left|e_\mu\right\rangle$, this proves that $\left|e_\mu(t)\right\rangle$ is simply a phase times $\left|e_\mu^0\right\rangle$:

$$\left|e_\mu(t)\right\rangle = e^{i\Phi(t)}\left|e_\mu^0\right\rangle \tag{34}$$

(to see this one can use the unicity of the solution of this differential equation given the initial conditions).

We will find the equation on $\Phi(t)$ by computing $\tilde{H}(t)\left|e_\mu^0\right\rangle$. Using that $\left|e_\mu^0\right\rangle$ is an eigenvector of $\tilde{H}_0$ and equation 33, we find :

$$\tilde{H}(t)\begin{pmatrix} u_\mu^0 \\ v_\mu^0 \end{pmatrix} = 2J\begin{pmatrix} u_\mu^0 \\ v_\mu^0 \end{pmatrix} + h_1 \sin(\Omega t)\begin{pmatrix} u_\mu^0 \\ -v_\mu^0 \end{pmatrix} \tag{35}$$

The important point to notice is that the second term (not proportional to $\left|e_\mu^0\right\rangle$) is small compared to the first one. To see this, we can take its scalar product with the other vectors of the basis (not forgetting the negative eigenvectors) :

$$\begin{aligned} |(u_\mu^0\ v_\mu^0)\cdot\begin{pmatrix} u_\nu^0 \\ -v_\nu^0 \end{pmatrix}| &= |u_\mu^0 \cdot u_\nu^0 - v_\mu^0 \cdot v_\nu^0| \leq 2 \\ |(v_\mu^0\ u_\mu^0)\cdot\begin{pmatrix} u_\nu^0 \\ -v_\nu^0 \end{pmatrix}| &= |v_\mu^0 \cdot u_\nu^0 - u_\mu^0 \cdot v_\nu^0| \leq 2 \end{aligned} \tag{36}$$

where we have used the Cauchy-Schwartz inequality together with the orthogonality conditions 17 to obtain the bounds. In any case, this scalar product is of order one, and so the second term can be neglected.

We have now proven the above statement, namely $\tilde{H}(t)\left|e_\mu^0\right\rangle = 2J\left|e_\mu^0\right\rangle$. Projecting on $\left\langle e_\mu^0\right|$, we find the evolution equation for Φ :

$$\frac{d}{dt}\Phi(t) = -4J \tag{37}$$

Integrating over one period, we find that $\Phi(T) = -4JT$. Given the definition of the Floquet spectrum as the spectrum of the evolution operator over one period defined in equation 28, we immediately see that all the Floquet eigenvalues are equal to $4J$ up to terms of order h_1, which is what we stated at the beginning of the Appendix.

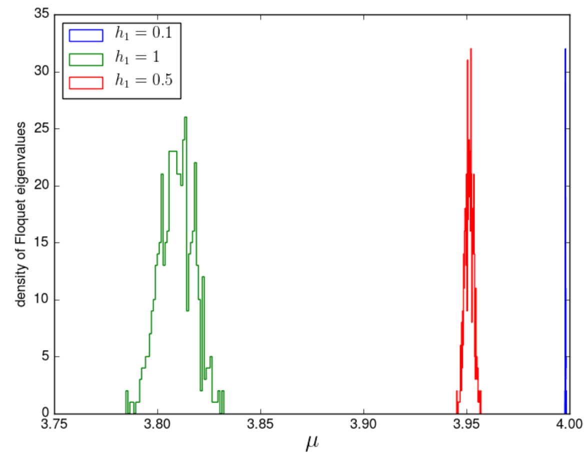


Figure 8: Floquet spectrum for parameters : $h_m = 0$, $h_0 = 0$, $L = 500$, $\gamma = 1$, $J = 1$, $T = 0.5$, and for different values of h_1. We observe that the peaked Floquet spectrum around $\mu = 4J$ disappears as we increase h_1, signaling that the second term in equation 35 is no longer small.

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